Star–in–Coloring of Some Splitting Graphs

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Abstract: A proper coloring of a graph \( G = (V, E) \) is a mapping \( f: V \to \{1,2,3,\ldots\} \) such that if \( e = v_i,v_j \in E \), then \( f(v_i) \neq f(v_j) \). A proper coloring of a directed graph (digraph) \( G \) is said to admit star–in–coloring, if the graph has to satisfy the following two conditions: (i) no path of length three is bicolored (ii) if any path of length twowith terminal vertices are of the same color, then the edges must be oriented towards the middle vertex. The minimum number of colors required to color a graph \( G \) as star–in–coloring, is called the star–in–chromatic number of the graph \( G \) and it is denoted by \( \chi_{si}(G) \). In this paper, we consider the splitting graphs of the graphs such as cycle, gear, regular cyclic, a complete binary tree and web graph, we investigate the star–in–chromatic number of these graphs.

Keywords: Coloring; Splitting graph; Star–in–coloring; Star–in–chromatic number.

1. INTRODUCTION

Let \( G = (V, E) \) be a simple, connected digraph with vertex set \( V = \{v_1,v_2,\ldots,v_n\} \) and edge set \( E \), each element of \( E \) is a directed edge. An orientation of a graph \( G \) is obtained by applying an orientation for each edge \( e = v_i,v_j \in E \) either from \( v_i \) to \( v_j \) or \( v_j \) to \( v_i \). A proper coloring of a graph \( G \) is a mapping \( f: V \to \{1,2,3,\ldots\} \) such that if \( e = v_i,v_j \in E \), then \( f(v_i) \neq f(v_j) \). A star–coloring of a graph \( G \) is a proper coloring of the graph with the condition that no path of length three \( (P_3) \) is bicolored. The concept of star–coloring of graphs was introduced by Grunbaum [2]. The star–coloring of graphs have been investigated by Fertin et al. [1] and Nesetril et al. [3].

A digraph \( G \) is said to be in–coloring if any path of length two with end vertices are of same color, then the edges are always directed towards the middle vertex. Motivated by the concepts of star–coloring and in–coloring, Sudha and Kanniga [5,6] have introduced a new concept known as star–in–coloring of graphs. A graph \( G \) is said to admit star–in–coloring if it satisfies the following two conditions.

- No path of length three \( (P_3) \) is bicolored.
- If any path of length two \( (P_3) \) with end vertices are of the same color, then the edges of \( P_3 \) are directed towards the middle vertex.

Sugumaran and Kasirajan [7] have found the lower, upper bounds and star–in–chromatic number of the graphs such as cycle, regular cyclic, gear, fan, double fan, web and complete binary tree.

Definition 1.1 The minimum number of colors required for the star–in–coloring of a graph \( G \) is called the star–in–chromatic number of \( G \) and it is denoted by \( \chi_{si}(G) \). The simplest star–in–coloring of a cycle \( C_4 \) is shown in Fig.1.

![Fig1. Star – in – coloring of Cycle C₄](image)

The star–in–chromatic number of the above graph is 3.
Definition 1.2 A cycle \( C_n \) is said to be a regular cyclic graph if maximum number of chords are drawn without forming a triangle and the resulting graph is regular. Then this regular cyclic graph is denoted by \( RC(p, n) \), where \( n \) is the degree of each vertex in this graph.

Definition 1.3 A connected acyclic graph is called a tree. A binary tree is a tree in which only one vertex is of degree two and each of the remaining vertices are of degree one or three. A vertex of degree two in a binary tree is called its root vertex. In a binary tree, a vertex \( v \) is said to be at level \( l \) if \( v \) is at a distance \( l \) from the root vertex.

Definition 1.4 A binary tree with level \( n \) is said to be complete if each level \( l \) of the binary tree contains exactly \( 2^l \) vertices, where \( 0 \leq l \leq n \). A complete binary tree with level \( n \) is denoted by \( BT_n \).

Note that the complete binary tree \( BT_n \) contains \( |V| = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \) vertices and \( |E| = |V| - 1 \) edges.

Definition 1.5 For any graph \( G \), the splitting graph \( S(G) \) is obtained by adding to each vertex \( v_i \) in \( G \) a new vertex \( v'_i \) such that \( N(v_i) = N(v'_i) \), where \( N(v_i) \) is the set of neighbours of the vertex \( v_i \).

Splitting graph was defined by Sampathkumar and Walikar [4].

2. MAIN RESULTS

In this section, we find the lower and upper bounds of the star–in–chromatic number of splitting graphs of some standard graphs. First we find out the star–in–chromatic number of a splitting graph of cycle \( C_n \).

Theorem 1. The graph \( S(C_n) \) admits star–in–coloring and its star–in–chromatic number satisfies the inequality \( 5 \leq \chi_{si}(S(C_n)) \leq 7 \), where \( n \) is even and \( n \geq 4 \).

Proof. Let \( G = S(C_n) \) and let \( V, E \) be the vertex set and edge set of \( G \) respectively. Then \( |V| = 2n \) and \( |E| = 3n \). For each \( i = 1, 2, ..., n \), let \( v_i \) be a vertex of cycle \( C_n \) and let \( v'_i \) be a new vertex (corresponding to a vertex \( v_i \) ) added in \( G \).

We define a function \( f: V \to \{1, 2, 3, ...\} \) such that \( f(v_i) \neq f(v'_j) \) if \( v_i, v'_j \in E \).

The star–in–coloring pattern of \( G \) we need to consider two different cases:

Case 1: For \( n \equiv 0 \pmod{4} \)

In a cycle \( C_n \), a path of length 3 is not bi-colored. So we need at least three colors.

\[
\begin{align*}
    f(v_i) &= \begin{cases} 
        1, & \text{if } i \equiv 1 \pmod{2} \\
        2, & \text{if } i \equiv 2 \pmod{4} \\
        3, & \text{if } i \equiv 0 \pmod{4}
    \end{cases}
\end{align*}
\]

Further,

\[
\begin{align*}
    f(v'_i) &= \begin{cases} 
        1, & \text{if } i \equiv 1 \pmod{2} \\
        4, & \text{if } i \equiv 2 \pmod{4} \\
        5, & \text{if } i \equiv 0 \pmod{4}
    \end{cases}
\end{align*}
\]

Case 2: For \( n \equiv 2 \pmod{4} \)

We assign \( f(v_0) = 6 \)

\[
\begin{align*}
    f(v_i) &= \begin{cases} 
        1, & \text{if } i \equiv 1 \pmod{2} \\
        2, & \text{if } i \equiv 2 \pmod{4} \\
        3, & \text{if } i \equiv 0 \pmod{4} \text{ and } i > 0
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    f(v'_i) &= \begin{cases} 
        1, & \text{if } i \equiv 1 \pmod{2} \\
        4, & \text{if } i \equiv 2 \pmod{4} \\
        5, & \text{if } i \equiv 0 \pmod{4} \text{ and } i > 0
    \end{cases}
\end{align*}
\]

From the above cases, we conclude that \( \chi_{si}(S(C_n)) = \begin{cases} 
5, & \text{if } n \equiv 0 \pmod{4} \\
7, & \text{if } n \equiv 2 \pmod{4}
\end{cases} \)
**Illustration:** The star–in–coloring of \( S(C_n) \) is shown for \( n = 6, 8 \) in Fig. 2 and Fig. 3 respectively. Note that the star–in–chromatic number \( \chi_{si}[S(C_6)] = 7 \) and \( \chi_{si}[S(C_8)] = 5 \). In general \( \chi_{si}[S(C_{4n})] = 5 \) and \( \chi_{si}[S(C_{4n+2})] = 7, n \in \mathbb{N} \).

**Theorem 2.** The splitting graph of Gear graph \( G_n \) admits star–in–coloring and its star–in–chromatic number satisfies the inequality \( 7 \leq \chi_{si}[S(G_n)] \leq 9 \), where \( n \geq 3 \).

**Proof.** Let \( G = S(G_n) \) and let \( V, E \) be the vertex set and edge set of \( G \) respectively. Then \( |V| = 4n + 2 \) and \( |E| = 9n \). For each \( i = 1, 2, \ldots, n \), let \( v_i \) be a vertex of Gear \( G_n \) and let \( v'_i \) be a new vertex (corresponding to the vertex \( v_i \)) added in \( G \).

We define a function \( f: V \to \{1, 2, 3, \ldots\} \) such that \( f(v_i) \neq f(v_j) \) if \( v_i, v_j \in E \).

The star–in–coloring pattern of \( G \) we need to consider two different cases:

**Case 1:** For \( n \) is even

We assign \( f(v_0) = 6, f(v'_0) = 7 \) and

\[
 f(v_i) = \begin{cases} 
 1, & \text{if } i \equiv 1 \pmod{2} \\
 2, & \text{if } i \equiv 2 \pmod{4} \\
 3, & \text{if } i \equiv 0 \pmod{4} \text{ and } i > 0 
\end{cases}
\]

\[
 f(v'_i) = \begin{cases} 
 1, & \text{if } i \equiv 1 \pmod{2} \\
 4, & \text{if } i \equiv 2 \pmod{4} \\
 5, & \text{if } i \equiv 0 \pmod{4} \text{ and } i > 0 
\end{cases}
\]

**Case 2:** For \( n \) is odd

We assign \( f(v_0) = 8, f(v'_0) = 9 \) and

\[
 f(v_i) = \begin{cases} 
 1, & \text{if } i \equiv 1 \pmod{2} \\
 2, & \text{if } i \equiv 2 \pmod{4} \text{ and } i < 2n - 2 \\
 3, & \text{if } i \equiv 0 \pmod{4} \text{ and } i > 0 \\
 4, & \text{if } i = 2n - 2. 
\end{cases}
\]
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\[ f(v_i) = \begin{cases} 
1, & \text{if } i \equiv 1 \text{ (mod 2)} \\
5, & \text{if } i \equiv 2 \text{ (mod 4)} \text{ and } i < 2n - 2 \\
6, & \text{if } i \equiv 0 \text{ (mod 4)} \text{ and } i > 0 \\
7, & \text{if } i = 2n - 2.
\]

From the above cases, we conclude that \( \chi_{si}[S(G_n)] = \begin{cases} 
7, & \text{if } n \text{ is even} \\
9, & \text{if } n \text{ is odd}
\end{cases} \)

**Illustration:** The star–in–coloring of \( S(G_n) \) is shown for \( n = 3, 4 \) in Fig. 4 and Fig. 5 respectively. Note that the star–in–chromatic number \( \chi_{si}[S(G_3)] = 9 \) and \( \chi_{si}[S(G_4)] = 7 \). In general \( \chi_{si}[S(G_{2n+2})] = 7 \) and \( \chi_{si}[S(G_{2n+1})] = 9, n \in N \).

**Theorem 3.** The splitting graph of a regular cyclic graph \( RC(p, n) \) admits star–in–coloring and its star–in–chromatic number is \( 2n + 1 \), where \( p \) is an even integer and \( p > 3 \).

**Proof.** Let \( G = S(RC(p, n)) \) and let \( V, E \) be the vertex set and edge set of \( G \) respectively. Then \( |V| = 4n \) and \( |E| = 3n^2 \). For each \( i = 1, 2, \ldots, n \), let \( v_i \) be a vertex of cycle \( G_n \) and let \( v' \) be a new vertex (corresponding to the vertex \( v_i \)) added in \( G \).

We define a function \( f: V \rightarrow \{1, 2, 3, \ldots\} \) such that \( f(v_i) \neq f(v_j) \) if \( v_i, v_j \in E \).

The star–in–coloring pattern is as follows:

We assign,
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\[ f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{2} \\ \frac{i}{2} + 1, & \text{if } i \equiv 2 \pmod{2} \end{cases} \]

\[ f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{2} \\ \frac{i}{2} + n + 1, & \text{if } i \equiv 2 \pmod{2} \end{cases} \]

By using the above pattern of coloring the splitting graph of regular cyclic graph is star–in–colored and \( \chi_{sl}[S(RC(p,n))] = 2n + 1 \).

**Illustration:** The star–in–coloring of \( S(RC(8,4)) \) is shown in Fig. 6. Note that the star–in–chromaticnumber \( \chi_{sl}[S(RC(8,4))] = 9 \).

**Fig 6. Splitting graph of RC(8,4)**

**Remark:** The graph \( RC(p,n) \) is not star–in–coloring, when \( p \) is odd, since at least one of the edges is left without orientation.

**Theorem 4.** The splitting graph of complete binary tree \( BT_n \) for all \( n \geq 2 \) admits star–in–coloring and \( \chi_{sl}[S(BT_n)] = 4 \).

**Proof.** Consider a complete binary tree \( BT_n \) with \( |V| = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \) vertices and \( |E| = |V| - 1 \) edges. The root vertex (degree 2) is denoted by \( v_0 \) and the other vertices are denoted by \( v_1, v_2, v_3, \ldots, v_n \).

The graph \( S(BT_n) \) consists of \( |V| = 2(1 + 2 + 2^2 + \cdots + 2^n) \) vertices and \( |E| = 3(|V| - 1) \) edges.

We define a function \( f: V \to \{ 1, 2, 3, \ldots \} \) such that \( f(v_i) \neq f(v_j) \) if \( v_i v_j \in E \).

The star–in–coloring pattern is as follows:

We assign \( f(v_0) = 1, f(u_0) = 3 \) and for each \( j = 1, 2, \ldots, 2^l \)

\[ f(v'_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod{3} \\ 2, & \text{if } i \equiv 1 \pmod{3} \\ 3, & \text{if } i \equiv 2 \pmod{3} \end{cases} \]

\[ f(u'_i) = 4, \text{ for all } 1 \leq i \leq n. \]

Note that \( f(u'_1) \neq 1 \) and 3, since by definition of proper coloring. Also if we assign \( f(u'_1) = 2 \), then the star – coloring condition for the path connecting the vertices \( u'_1, v'_2, v'_1, v'_2 \) is affected. Hence a new color 4 is assigned to the vertex \( u'_1 \).

With this pattern of coloring, the splitting graph of complete binary tree \( BT_n \) can be star–in–colored and \( \chi_{sl}[S(BT_n)] = 4 \) for all \( n \geq 2 \).

**Illustration:** The star–in–coloring of \( S(BT_2) \) is shown in Fig. 7. Note that the star–in–chromaticnumber \( \chi_{sl}[S(BT_2)] = 4 \).
Theorem 5. The splitting graph of the web graph $W_{n,r}$ admits star–in–coloring and its star–in–chromatic
coloring number satisfies the inequality $9 \leq \chi_{si}(S(W_{n,r})) \leq 11$, for all even $n$.

Proof. The graph $S(W_{n,r})$ consists of $2nr$ vertices and $3n(2r - 1)$ edges. The vertex sets $V$ and $U$
in $W_{n,r}$ are partitioned into vertex sets denoted by $V^1, V^2, V^3, ..., V^r$ and $U^1, U^2, U^3, ..., U^r$ where each
vertex set consists of $n$ vertices. Assume that the vertex set $V^j$ consists of the vertices $v^j_1, v^j_2, ..., v^j_n$ and the vertex set $U^j$ consists of the vertices $u^j_1, u^j_2, ..., u^j_n$ for all $1 \leq j \leq r$.

The general pattern of coloring has been grouped into two cases: One for $n \equiv 0 \pmod{4}$ and other
for $n \equiv 2 \pmod{4}$.

Case 1: For $n \equiv 0 \pmod{4}$

$f(v^1_i) = 1$ and $f(u^1_i) = 1$ if $i + j$ is even.

For all other values of $i$ and $j$, we consider four subcases as follows:

Subcase 1.1: For $j \equiv 1 \pmod{4}$

$f(v^1_i) = \begin{cases} 2, & \text{if } i \equiv 2 \pmod{4} \\ 3, & \text{if } i \equiv 0 \pmod{4} \end{cases}$

$f(u^1_i) = \begin{cases} 6, & \text{if } i \equiv 2 \pmod{4} \\ 7, & \text{if } i \equiv 0 \pmod{4} \end{cases}$

Subcase 1.2: For $j \equiv 2 \pmod{4}$

$f(v^1_i) = \begin{cases} 4, & \text{if } i \equiv 1 \pmod{4} \\ 5, & \text{if } i \equiv 3 \pmod{4} \end{cases}$

$f(u^1_i) = \begin{cases} 8, & \text{if } i \equiv 1 \pmod{4} \\ 9, & \text{if } i \equiv 3 \pmod{4} \end{cases}$

Subcase 1.3: For $j \equiv 3 \pmod{4}$

$f(v^1_i) = \begin{cases} 3, & \text{if } i \equiv 2 \pmod{4} \\ 2, & \text{if } i \equiv 0 \pmod{4} \end{cases}$

$f(u^1_i) = \begin{cases} 7, & \text{if } i \equiv 2 \pmod{4} \\ 6, & \text{if } i \equiv 0 \pmod{4} \end{cases}$

Subcase 1.4: For $j \equiv 0 \pmod{4}$
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\[
\begin{align*}
f(v_i^j) &= \begin{cases} 5, & \text{if } i \equiv 1 \pmod{4} \\ 4, & \text{if } i \equiv 3 \pmod{4} \end{cases} \\
f(u_i^j) &= \begin{cases} 9, & \text{if } i \equiv 1 \pmod{4} \\ 8, & \text{if } i \equiv 3 \pmod{4} \end{cases}
\end{align*}
\]

By using the above pattern of coloring the web graph is star–in–colored. According to Case 1 the star–in–chromatic number of \(W_{n,r}\) is \(\chi_{si}(W_{n,r}) = 9\).

**Case 2:** For \(n \equiv 2 \pmod{4}\)

\[
\begin{align*}
f(v_i^j) &= 1 \text{ and } f(u_i^j) = 1 \text{ if } i + j \text{ is even.}
\end{align*}
\]

For all other values of \(i\) and \(j\), we consider four subcases as follows:

**Subcase 2.1:** For \(j \equiv 1 \pmod{4}\)

\[
\begin{align*}
f(v_i^j) &= \begin{cases} 2, & \text{if } i \equiv 2 \pmod{4} \text{ and } i < n \\ 3, & \text{if } i \equiv 0 \pmod{4} \\ 4, & \text{if } i = n \end{cases} \\
f(u_i^j) &= \begin{cases} 8, & \text{if } i \equiv 2 \pmod{4} \text{ and } i < n \\ 9, & \text{if } i \equiv 0 \pmod{4} \\ 10, & \text{if } i = n \end{cases}
\end{align*}
\]

**Subcase 2.2:** For \(j \equiv 2 \pmod{4}\)

\[
\begin{align*}
f(v_i^j) &= \begin{cases} 5, & \text{if } i \equiv 1 \pmod{4} \text{ and } i < n - 1 \\ 6, & \text{if } i \equiv 3 \pmod{4} \\ 7, & \text{if } i = n - 1 \end{cases} \\
f(u_i^j) &= \begin{cases} 11, & \text{if } i \equiv 1 \pmod{4} \text{ and } i < n - 1 \\ 12, & \text{if } i \equiv 3 \pmod{4} \\ 13, & \text{if } i = n - 1 \end{cases}
\end{align*}
\]

**Subcase 2.3:** For \(j \equiv 3 \pmod{4}\)

\[
\begin{align*}
f(v_i^j) &= \begin{cases} 4, & \text{if } i \equiv 2 \pmod{4} \text{ and } i < n \\ 2, & \text{if } i \equiv 0 \pmod{4} \\ 3, & \text{if } i = n \end{cases} \\
f(u_i^j) &= \begin{cases} 10, & \text{if } i \equiv 2 \pmod{4} \text{ and } i < n \\ 8, & \text{if } i \equiv 0 \pmod{4} \\ 9, & \text{if } i = n \end{cases}
\end{align*}
\]

**Subcase 2.4:** For \(j \equiv 0 \pmod{4}\)

\[
\begin{align*}
f(v_i^j) &= \begin{cases} 7, & \text{if } i \equiv 1 \pmod{4} \text{ and } i < n - 1 \\ 5, & \text{if } i \equiv 3 \pmod{4} \\ 6, & \text{if } i = n - 1 \end{cases} \\
f(u_i^j) &= \begin{cases} 13, & \text{if } i \equiv 1 \pmod{4} \text{ and } i < n - 1 \\ 11, & \text{if } i \equiv 3 \pmod{4} \\ 12, & \text{if } i = n - 1 \end{cases}
\end{align*}
\]

From the above cases, we conclude that \(\chi_{si}[S(W_{n,r})] = \begin{cases} 9, & \text{if } n \equiv 0 \pmod{4} \\ 13, & \text{if } n \equiv 2 \pmod{4} \end{cases}\).

**Illustration:** The star–in–coloring of \(S(W_{n,r})\) for \(n = 4, 6\) and \(r = 4\) are shown in Fig. 8 and Fig. 9 respectively. Note that the star–in–chromatic number \(\chi_{si}[S(W_{4,4})] = 9\) and \(\chi_{si}[S(W_{6,4})] = 13\). In general \(\chi_{si}[S(W_{n,r})] = 9\) and \(\chi_{si}[S(W_{n+2,r})] = 13, n \in N\).
3. CONCLUSION

In this paper, we have shown that the lower and upper bounds for star–in–chromatic number of some of the standard graphs are as given below.

1. $5 \leq \chi_{si}[S(C_n)] \leq 7$ if $n$ is even.
2. $7 \leq \chi_{si}[S(G_n)] \leq 9$.
3. $2n + 1, p > 3$.
4. $\chi_{si}[S(BT_n)] = 4, n \geq 2$.
5. $9 \leq \chi_{si}[S(W_{n,r})] \leq 13$. 

Fig 8. Splitting graph of $W_{4,4}$

Fig 9. Splitting graph of $W_{6,4}$
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