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Cyclic Contraction and Fixed-Point Theorem in b-Metric Space using Rational Inequalities

Raksha Dwivedi*

Research Scholar Department of Mathematic Sarvepalli Radhakrishnan University, Bhopal (M.P.), India

Dr. Richa Professor, Department of Mathematics Sarvepalli Radhakrishnan University, Bhopal, India

***Corresponding Author:** **Raksha Dwivedi,** Research Scholar Department of Mathematic Sarvepalli Radhakrishnan University, Bhopal (M.P.), India*

Abstract: *In this paper, we prove a fixed-point theorem for cyclic contraction using rational inequality in b-metric space.*

Keywords: *Cyclic Contraction, Cauchy Sequence, b-metric space, Fixed-point.*

1. Introduction

In 1989, Backhtin [1] introduced the concept of b-metric space. In 1993, Czerwik [5] extended the results of b-metric spaces. Using this idea many researchers presented generalization of the renowned Banach fixed-point theorem in the b-metric space.

Definition 1.1:-

Let X be a non-empty set $k \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{R}_+$ is called a b-metric provided that $\forall\, x, y, z \in X$

1) $d(x,y) = 0 \iff x = y,$
2) $d(x,y) = d(y,x),$
3) $d(x,z) \leq k\,[d(x,y) + d(y,z)].$

A pair (X, d) is called a b-metric space. It is clear that definition of b-metric space is an extension of usual metric space.

Example of b-metric space have given below:

Example (a) By Boriceanu [4], Let $X = \{0,1,2\}$ and $d(2,0) = d(0,2) = m \geq 2,$

$$d(0,1) = d(1,2) = d(1,0) = d(2,1) = 1$$

$$\text{and } d(0,0) = d(1,1) = d(2,2) = 0$$

$$\text{then } d(x,y) \leq \frac{m}{2}[d(x,z) + d(z,y)] \quad \forall x, y, z \in X.$$

if $m > 2$ then the triangle inequality does not hold.

2. Preliminaries

Definition 2.1 By Boriceanu [4], Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence $\iff \forall \epsilon > 0\ \exists\, n(\epsilon) \in N$ such that for each $n, m \geq n(\epsilon)$ we have $d(x_n, x_m) < \epsilon$.

Definition 2.2 Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called convergent sequence $\iff \exists\, x \in X$ such that $\forall\ \exists\, n(\epsilon) \in N$ such that $\forall\, n \geq n(\epsilon)$ we have $d(x_n, x) < \epsilon$, in this case we write $\lim_{n\to\infty} x_n = x$.

Definition 2.3 Let (X, d) be a b-metric space is complete if every Cauchy sequence convergent.

In 2003, Kirk et al. [12] introduced Cyclic contractions in metric spaces and investigated the existence of proximity points and fixed-points in view of cyclic contraction mappings as follows.

Definition 2.4 Let X be a metric space and A and B are closed subsets of X .Then Function $S: A \cup B \to A \cup B$ is said to be a Cyclic map if $S(A) \subset B$ and $S(B) \subset A$.

$$\forall x \in X, n = 0,1,2 \ldots \ldots \infty, \text{where } x \in A, S(x) \in B$$

$$S(Sx) = S^2x \in A \Longrightarrow S^{2n} \in A$$

$$S(S^2x) = S^3x \in B \Longrightarrow S^{2n-1} \in B$$

$\Longrightarrow \{S^{2n}(x)\}$ is Sequence in A and $\{S^{2n-1}(x)\}$ is Sequence in B. S is called Cyclic mapping. Then any contraction is called Cyclic Contraction

3. Main Result

Theorem 3.1 Let (X, d) be a complete b-metric space and A and B are closed non $-$ empty subsets of X. $S: A \cup B \to A \cup B$ be a contraction mapping satisfying the following contraction.

$$d(Sx, Sy) \leq \alpha . \frac{d(Sx, x)[1 + d(Sy, y)]}{1 + d(x, y)} + \beta . d(x, y)$$

$\forall x \in A, y \in B, \alpha, \beta > 0$ and $\alpha + \beta < 1$. Then S has a unique fixed $-$ point in $A \cap B$.

Proof:- Let $x \in A, Sx \in B$, $S^2x \in A, S^3x \in B$ in general $S^{2n}x \in A$, $S^{2n-1}x \in B$ $\{S^nx\}$ is a sequence in X.

$$d(S^2x, Sx) = d(S.(Sx), Sx)$$

$$\leq \alpha . \frac{d(S.Sx, Sx)[1 + d(Sx, x)]}{1 + d(Sx, x)} + \beta . d(Sx, x)$$

$$d(S^2x, Sx) \leq \alpha\, d(S.Sx, Sx) + \beta . d(Sx, x)$$

$$d(S^2x, Sx)(1 - \alpha) \leq \beta . d(Sx, x)$$

$$d(S^2x, Sx) \leq \frac{\beta}{(1 - \alpha)} . d(Sx, x)$$

$$\text{Where } h = \frac{\beta}{(1 - \alpha)} < 1$$

$$d(S^2x, Sx) \leq h\, d(Sx, x)$$

Similarly,

$$d(S^3x, S^2x) \leq h^2 d(Sx, x)$$

$$d(S^{n+1}x, S^nx) \leq h^{n-1} d(Sx, x)$$

Let $m, n \in N$ and $m > n$ using triangular inequality we have

$$d(S^mx, S^nx) \leq k^{m-n} d(S^mx, S^{m-1}x) + k^{m-n-1} d(S^{m-1}x, S^{m-2}x) + \cdots \ldots . . + k\, d(S^{n+1}x, S^nx)$$

$$d(S^mx, S^nx) \leq (k^{m-n}h^{m-1} + k^{m-n-1}h^{m-2} + \cdots \ldots + k\, h^n) d(Sx, x)$$

Further simplification minimizes to

$$d(S^mx, S^nx) \leq [(kh)^{m-n} . h^{n-1} + (kh)^{m-n-1} . h^{n-1} + \cdots \ldots + kh\, . h^{n-1}] d(Sx, x)$$

$$d(S^mx, S^nx) \leq [h^{n-1} + h^{n-1} + \cdots \ldots . . + h^{n-1}]\, d(Sx, x)$$

$$= h^{n-1} (m - n - 1) d(Sx, x)$$

$$\leq h^{n-1}\, \delta\, d(Sx, x)$$

With $\delta > 0$ and as $n \to \infty, kh < 1$ we get $d(S^m x, S^n x) \to 0$

Therefore $\{S^n x\}$ is a Cauchy sequence in X

$\{S^n x\}$ converges to $u \in X$ as (X, d) is complete.

Sequence $\{S^n x\}$ is in A and Sequence $\{S^{n-1} x\}$ is in B in such a way that both converges

to $u \in X$.

as A and B are closed subsets of X . Hence $u \in A \cap B$ and $A \cap B \neq \emptyset$

Now we prove $Su = u$

We have $d(S^n x, Su) = d(S.S^{n-1}x, Su)$

$$\leq \alpha. \frac{d(S.S^{n-1}x, S^{n-1}x)[1 + d(Su, u)]}{1 + d(S^{n-1}x, u)} + \beta\, d(S^{n-1}x, u)$$

$$\leq \alpha. \frac{d(S^n x, S^{n-1}x)[1 + d(Su, u)]}{1 + d(S^{n-1}x, u)} + \beta\, d(S^{n-1}x, u)$$

Taking $n \to \infty$

$$d(u, Su) \leq \alpha. \frac{d(u, u)[1 + d(Su, u)]}{1 + d(u, u)} + \beta\, d(u, u)$$

$d(u, Su) = 0 \implies \quad Su = u$

Now we establish uniqueness.

Let $v \in X$ be other fixed – point of S.

Then $Sv = v$

We have $d(u, v) = d(Su, Sv)$

$$\leq \alpha. \frac{d(Su, u)[1 + d(Sv, v)]}{1 + d(u, v)} + \beta\, d(u, v)$$

$$\leq \alpha. \frac{d(u, u)[1 + d(u, v)]}{1 + d(u, v)} + \beta\, d(u, v)$$

$d(u, v)(1 - \beta) \leq 0$

$\because (1 - \beta) \neq 0$

Hence $\quad u = v$

This completes the prove of the Theorem.

Theorem 3.2 Let (X, d) be a complete b-metric space and A and B are closed non – empty subsets of X. $S: A \cup B \to A \cup B$ be a contraction mapping satisfying the following contraction.

$$d(Sx, Sy) \leq \alpha. \frac{d(x, Sx)\,.\,d(y, Sy)}{d(x, y)} + \beta.\, d(x, y)$$

$\forall x \in A, y \in B, \alpha, \beta > 0$ and $\alpha + \beta < 1$. Then S has a unique fixed – point in $A \cap B$.

Proof:- Let $x \in A, Sx \in B, \quad S^2 x \in A, S^3 x \in B$ in general $S^{2n} x \in A,\ S^{2n-1} x \in B$ $\{S^n x\}$ is a sequence in X.

$d(S^2 x, Sx) = d(S.(Sx), Sx)$

$$\leq \alpha. \frac{d(Sx, S.Sx)\,.\,d(x, Sx)}{d(Sx, x)} + \beta.\, d(Sx, x)$$

$d(S^2x, Sx) \leq \alpha\, d(S.Sx, Sx) + \beta.d(Sx, x)$

$d(S^2x, Sx)(1-\alpha) \leq \beta.d(Sx, x)$

$d(S^2x, Sx) \leq \dfrac{\beta}{(1-\alpha)}.d(Sx, x)$

Where $h = \dfrac{\beta}{(1-\alpha)} < 1$

$d(S^2x, Sx) \leq h\, d(Sx, x)$

Similarly,
$d(S^3x, S^2x) \leq h^2 d(Sx, x)$

$d(S^{n+1}x, S^nx) \leq h^{n-1} d(Sx, x)$

Let $m, n \in N$ and $m > n$ using triangular inequality we have

$d(S^mx, S^nx) \leq k^{m-n} d(S^mx, S^{m-1}x) + k^{m-n-1} d(S^{m-1}x, S^{m-2}x) + \cdots \ldots .. + k\, d(S^{n+1}x, S^nx)$

$d(S^mx, S^nx) \leq (k^{m-n}h^{m-1} + k^{m-n-1}h^{m-2} + \cdots \ldots + k\, h^n) d(Sx, x)$

Further simplification minimizes to

$d(S^mx, S^nx) \leq [(kh)^{m-n}.h^{n-1} + (kh)^{m-n-1}.h^{n-1} + \cdots \ldots + kh\,.h^{n-1}] d(Sx, x)$

$d(S^mx, S^nx) \leq [h^{n-1} + h^{n-1} + \cdots \ldots .. + h^{n-1}]\, d(Sx, x)$

$= h^{n-1}\,(m-n-1) d(Sx, x)$

$\leq h^{n-1}\,\delta\, d(Sx, x)$

With $\delta > 0$ and as $n \to \infty, kh < 1$ we get $d(S^mx, S^nx) \to 0$

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Now we prove $Su = u$

We have $d\,(S^nx, Su) = d(S.S^{n-1}x, Su\,)$

$\leq \alpha.\dfrac{d(S^{n-1}x, S.S^{n-1}x).d(u, Su)}{d(S^{n-1}x, u)} + \beta\, d(S^{n-1}x, u)$

$\leq \alpha.\dfrac{d(S^{n-1}x, S^nx).d(u, Su)}{1 + d(S^{n-1}x, u)} + \beta\, d(S^{n-1}x, u)$

Taking $n \to \infty$

$d(u, Su) \leq \alpha.\dfrac{d(u, u).d(u, u)}{d(u, u)} + \beta\, d(u, u)$

$d(u, Su) = 0 \implies \quad Su = u$

Now we establish uniqueness.

Let $v \in X$ be other fixed − point of S.

Then $Sv = v$

We have $d(u, v) = d(Su, Sv)$

$\leq \alpha.\dfrac{d(Su, u).d(v, Sv)}{d(u, v)} + \beta\, d(u, v)$

$\leq \alpha.\dfrac{d(u, u).d(v, v)}{d(u, v)} + \beta\, d(u, v)$

$d(u, v)(1-\beta) \leq 0$

$\because (1-\beta) \neq 0$

Hence $\quad u = v$

This completes the prove of the Theorem.

Theorem 3.3 Let (X, d) be a complete b-metric space and A and B are closed non − empty subsets of X. $S: A \cup B \to A \cup B$ be a contraction mapping satisfying the following contraction.

$$d(Sx, Sy) \leq \alpha.\frac{\{d(x, Sx)\}^2}{d(Sx, x) + d(Sy, y)}$$

$\forall x \in A, y \in B, 0 < \alpha < 2$. Then S has a unique fixed − point in $A \cap B$.

Proof:- Let $x \in A, Sx \in B$, $S^2x \in A, S^3x \in B$ in general $S^{2n}x \in A$, $S^{2n-1}x \in B$ $\{S^n x\}$ is a sequence in X.

$$d(S^2x, Sx) = d(S.(Sx), Sx)$$

$$\leq \alpha.\frac{\{d(Sx, S.Sx)\}^2}{d(S.Sx, Sx) +. d(Sx, x)}$$

$$d(S^2x, Sx) \leq \alpha.\frac{\{d(Sx, S^2x)\}^2}{d(S^2x, Sx) +. d(Sx, x)}$$

$$d(S^2x, Sx) \leq \alpha.\frac{d(S^2x, Sx).d(S^2x, Sx)}{d(S^2x, Sx) +. d(Sx, x)}$$

$$\frac{1}{\alpha.d(S^2x, Sx)} \leq \frac{1}{d(S^2x, Sx) + d(Sx, x)}$$

$$\alpha.d(S^2x, Sx) \leq d(S^2x, Sx) + d(Sx, x)$$

$$d(S^2x, Sx)(\alpha - 1) \leq d(Sx, x)$$

$$d(S^2x, Sx) \leq \frac{1}{(\alpha - 1)}d(Sx, x)$$

Where $\quad h = \frac{1}{(\alpha-1)} < 2$

$$d(S^2x, Sx) \leq h\, d(Sx, x)$$

Similarly,

$$d(S^3x, S^2x) \leq h^2 d(Sx, x)$$

$$d(S^{n+1}x, S^n x) \leq h^{n-1} d(Sx, x)$$

Let $m, n \in N$ and $m > n$ using triangular inequality we have

$$d(S^m x, S^n x) \leq k^{m-n} d(S^m x, S^{m-1}x) + k^{m-n-1} d(S^{m-1}x, S^{m-2}x) + \cdots \ldots\ldots + k\, d(S^{n+1}x, S^n x)$$

$$d(S^m x, S^n x) \leq (k^{m-n} h^{m-1} + k^{m-n-1} h^{m-2} + \cdots \ldots + k\, h^n) d(Sx, x)$$

Further simplification minimizes to

$$d(S^m x, S^n x) \leq [(kh)^{m-n}.h^{n-1} + (kh)^{m-n-1}.h^{n-1} + \cdots \ldots + kh\,.h^{n-1}] d(Sx, x)$$

$$d(S^m x, S^n x) \leq [h^{n-1} + h^{n-1} + \cdots \ldots\ldots + h^{n-1}]\, d(Sx, x)$$

$$= h^{n-1}(m - n - 1) d(Sx, x)$$

$$\leq h^{n-1}\, \delta\, d(Sx, x)$$

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$\leq \alpha.\frac{\{d(S^{n-1}x, S.S^{n-1}x)\}^2}{d(S.S^{n-1}x, S^{n-1}x) + .d(Su, u)}$

$d(S^n x, Su) \leq \alpha.\frac{\{d(S^{n-1}x, S^n x)\}^2}{d(S^n x, S^{n-1}x) + .d(Su, u)}$

Taking $n \to \infty$

$d(u.Su) \leq \alpha.\frac{\{d(u, Su)\}^2}{d(u, Su) + d(u, u)}$

$\frac{1}{\alpha.d(u.Su)} \leq \frac{1}{d(u.Su) + d(u.u)}$

$\alpha.d(u.Su) \leq d(u.Su) + d(u.u)$

$d(Su, u)(\alpha - 1) \leq 0$

$\because (\alpha - 1) \neq 0$

$d(Su, u) = 0$

$Su = u$

Now we establish uniqueness.

Let $v \in X$ be other fixed $-$ point of S.

Then $Sv = v$

We have $d(u, v) = d(Su, Sv)$

$\leq \alpha.\frac{\{d(u, Su)\}^2}{d(Su, u) + d(v, Sv)}$

$d(u, v) \leq \alpha.\frac{\{d(u, Su)\}^2}{d(Su, u) + d(v, Sv)}$

$\leq \alpha.\frac{\{d(u, u)\}^2}{d(u, u) + d(v, v)}$

$d(u, v) \leq 0$

Or $\quad u = v$

This completes the prove of the Theorem.

Corollary:- Let (X, d) be a complete b-metric space and A and B are closed non $-$ empty subsets of X. $S: A \cup B \to A \cup B$ be a contraction mapping satisfying the following contraction.

$$d(Sx, Sy) \leq \alpha.\frac{\{d(x, Sx)\}^3}{d(Sx, Sy).[d(Sx, x) + d(y, Sy)]}$$

$\forall x \in A, y \in B, 0 < \alpha < 2$. Then S has a unique fixed $-$ point in $A \cap B$.

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