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Fixed Point Theorem on Complete Metric Space

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Abstract: *The aim of this paper we prove a simple result on complete metric space.*

Keywords: *Fixed-point, Complete metric Space*

1. Introduction

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed-point theory is one of the most important tools.

2. Preliminaries

Definition 2.1 : - A Sequence $\{x_n\}$ in Complete metric space

(X, d) is called Cauchy sequence if given $\epsilon > 0$, $\exists$ $n_0 \in$ N such that $\forall\, m, n \geq n_0$

$d(x_m, x_n) < \epsilon$ or $d(x_m, x_n) < \epsilon$

i.e. $\min\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$

Definition 2.2 : - A Sequence $\{x_n\}$ in Complete metric space converges to x if

$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$ and x is called limit of $\{x_n\}$.

Theorem 1: Let F and G be mappings of a complete metric space (X, d) into $B(X)$ Satisfying the inequality

$$\delta(Fx, Gy) \leq c.\max\{\delta(x, Fx), \delta(y, Gy), d(x, y)\}$$

$\forall\, x, y \in X$ Where $0 \leq c < 1$ then F and G have a common unique fixed point.

Inspired by above theorem we prove following theorem. (in main result)

3. Main Result

Theorem 3.1 Let (X, d) be a complete metric space. Let S is a continuous mapping $S: X \to X$ such that

$$d^2(Sx, Sy) \leq c.\max\{d(x,y).d(x,Sx), d(x,y).d(y,Sy), d(x,y).d(x,Sy),$$
$$d(x,Sx).d(y,Sy), d(x,Sx).d(x,Sy), d(y,Sy).d(x,Sy)\}$$

Where $c \in (0,1)$ and $x, y \in X$ *and* $s \geq 1$ *Then* S has a unique fixed-point.

Proof: Let $x_0 \in X$ be any arbitrary in X and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.Defined by the recursion. Let

$S(x_0) = x_1$, $S(x_1) = x_2$, in general.

$S(x_{n-1}) = x_n$, $S(x_{n+1}) = x_{n+1}$, $n = 0,1,2,3 \ldots\ldots\ldots$

$$d^2(x_{n+1}, x_n) = d^2(Sx_n, Sx_{n-1})$$
$$\leq c.\max\{d(x_n, x_{n-1}).d(x_n, Sx_n), d(x_n, x_{n-1}).d(x_{n-1}, Sx_{n-1}), d(x_n, x_{n-1}).d(x_n, Sx_{n-1}),$$
$$d(x_n, Sx_n).d(x_{n-1}, Sx_{n-1}), d(x_n, Sx_n).d(x_n, Sx_{n-1}), d(x_{n-1}, Sx_{n-1}).d(x_n, Sx_{n-1})\}$$

$d^2(x_{n+1}, x_n)$

$\leq c.\max\{\, d(x_n, x_{n-1}).d(\, x_n, x_{n+1}\,),\ d(x_n, x_{n-1}).d(x_{n-1}, x_n), d(x_n, x_{n-1}).d(x_n, x_n),$

$d(x_n, x_{n+1}).d(x_{n-1}, x_n), d(x_n, x_{n+1}).d(x_n, x_n), d(x_{n-1}, x_n).d(x_n, x_n)\}$

$d^2(x_{n+1}, x_n)\ \leq c.\max\{\, d(x_n, x_{n-1}).d(\, x_n, x_{n+1}\,),\ d(x_n, x_{n-1}).d(x_{n-1}, x_n), d(x_n, x_{n-1}).\ 0,$

$d(x_n, x_{n+1}).d(x_{n-1}, x_n), d(x_n, x_{n+1}).\ 0, d(x_{n-1}, x_n).\ 0\}$

$d^2(x_{n+1}, x_n)\ \leq c.\max\{\, d(x_n, x_{n-1}).d(\, x_n, x_{n+1}\,),\ d(x_n, x_{n-1}).d(x_{n-1}, x_n),$

$d(x_n, x_{n+1}).d(x_{n-1}, x_n),\}$

$d^2(x_{n+1}, x_n)\ \leq c.\max\{\, d(x_n, x_{n-1}).d(\, x_n, x_{n+1}\,),\ d^2(x_n, x_{n-1}),\}$

$d^2(x_{n+1}, x_n)\ \leq c\,.\,M_1$

$Where\ \ M_1 = \max\{\, d(x_n, x_{n-1}).d(\, x_n, x_{n+1}\,),\ d^2(x_n, x_{n-1}),\}$

$Now\ two\ cases\ arises$

$Case\ I:\ If\ suppose\ that\ \ \ M_1 = d(x_n, x_{n-1}).d(\, x_n, x_{n+1}\,)$

$d^2(x_{n+1}, x_n)\ \leq c.d(x_n, x_{n-1}).d(\, x_n, x_{n+1}\,)$

$d(x_{n+1}, x_n)\ \leq c.d(x_n, x_{n-1})$

$Where\ \ \ c = k \leq 1$

$d(x_{n+1}, x_n)\ \leq k\,.d(x_n, x_{n-1})$

$d(x_{n+1}, x_n)\ \leq k^2.d(x_{n-1}, x_{n-2})$

$Continuing\ this\ process$

$d(x_{n+1}, x_n)\ \leq\ k^n.d(x_1, x_0)$

$Case\ II:\ If\ suppose\ that\ \ \ M_1 =\ d^2(x_n, x_{n-1})$

$d^2(x_{n+1}, x_n)\ \leq c.d^2(x_n, x_{n-1})$

$d(x_{n+1}, x_n)\ \leq \sqrt{c}.d(x_n, x_{n-1})$

$Where\ \ \ h \leq \sqrt{c}$

$d(x_{n+1}, x_n)\ \leq h\,.d(x_n, x_{n-1})$

$d(x_{n+1}, x_n)\ \leq h^2.d(x_{n-1}, x_{n-2})$

$Continuing\ this\ process$

$d(x_{n+1}, x_n)\ \leq\ h^{n+1}.d(x_1, x_0)$

$It\ follows\ that$

$d(x_n, x_{n+r}\,)\ \leq\ \ d(x_n, x_{n+1}) + d(x_{n+1}, x_{n-2}) +\ ------- + d(x_{n+r-1}, x_{n+r})$

$\leq (h^n +\ ------- + h^{n+r-1}).d(x_0, x_1)$

$\leq \frac{h^n}{1-h}\,.d(x_0, x_1)$

$Since\ h < 1\ \ \ \ \ and\ \ \epsilon > 0\ \ , \exists\ n_0 \in N$

$Such\ that\ d(x_m, x_n) \leq \epsilon\ \ \ \ \ , \forall\ m, n\ \ \geq n_0$

$It\ follows\ that\ \ \ \{x_n\}_{n=1}^{\infty}\ \ is\ a\ Cauchy\ Sequence\ converges\ to\ some\ x\ , x \in X, as\ X\ is\ complete.$

Now

$d(x, x_n) \leq\ \ d(x, x_m) + d(x_m, x_n)$

$\leq\ d(x, x_m) + \epsilon\ \ \ \ , \forall\ \ m, n \geq\ \ n_0$

$Let\ m \to \infty$

$d(x, x_n) \leq \ \epsilon \quad , \forall \ \ n \geq \ n_0$

$Thus$

$d^2(Sx, Sx_{n-1})$

$\leq c.\max\{\ d(x, x_{n-1}).\ d(\ x, Sx_n),\ d(x, x_{n-1}).\ d(x_{n-1}, Sx_{n-1}), d(x, x_{n-1}).\ d(x, Sx_{n-1}),$

$\ d(x, Sx).\ d(x_{n-1}, Sx_{n-1}), d(x, Sx).\ d(x, Sx_{n-1}), d(x_{n-1}, Sx_{n-1}).\ d(x, Sx_{n-1})\}$

$d^2(Sx, Sx_{n-1})$

$\leq c.\max\{\ d(x, x_{n-1}).\ d(\ x, Sx),\ d(x, x_{n-1}).\ d(x_{n-1}, x_n), d(x, x_{n-1}).\ d(x, x_n),$

$\ d(x, Sx).\ d(x_{n-1}, x_n), d(x, Sx).\ d(x, x_n), d(x_{n-1}, x_n).\ d(x, x_n)\}$

$Taking\ \ n \to \infty$

$d^2(Sx, Sx_{n-1}) = \ \ d^2(Sx, x_n\ ) = d^2(Sx, x) \leq c.\{0\} = 0$

$d^2(Sx, x) = 0$

$d(Sx, x) = 0$

$Sx = x$

4. Uniqueness of Fixed-Point

$Assume\ that\ y\ is\ another\ fixed - point\ of\ X. Then\ we\ have$

$Sy = y \quad and$

$d^2(x, y) = d^2(Sx, Sy) \leq \ \ c.\max\{d(x, y).\ d(x, Sx), d(x, y).\ d(y, Sy), d(x, y).\ d(x, Sy),$

$d(x, Sx).\ d(y, Sy), d(x, Sx).\ d(x, Sy), d(y, Sy).\ d(x, Sy)\}$

$d^2(x, y)\ \leq\ c\ .max\{d(x, y).\ d(x, x), d(x, y).\ d(y, y), d(x, y).\ d(x, y),$

$d(x, x).\ d(y, y), d(x, x).\ d(x, y), d(y, y).\ d(x, y)\}$

$d^2(x, y)\ \leq\ c\ .max\{0,0, d(x, y).\ d(x, y), 0,0,0\}$

$d^2(x, y)\ \leq\ c\ .d(x, y).\ d(x, y),$

$d(x, y)\ \leq\ c\ .d(x, y)$

$This\ is\ contradiction\ .Therefore$

$x = y$

$This\ completes\ the\ proof\ .Hence\ x\ is\ the\ unique\ fixed - point\ of\ S.$

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