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Discrete – time State Space Linear Optimum Control

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Abstract: *The paper presents new methods for state space deterministic and stochastic linear control design for systems with constant parameters. Principle of the methods is optimization on a section of two control stages. The deterministic finite horizon optimum control problem is solved through a sequence of two-stage optimizations and iteration. Solution of the infinite horizon optimum control problem is derived from optimization on two stages and limit value calculation, without iteration. The same principles govern all the solutions for deterministic and stochastic control.*

Keywords: *Linear Quadratic Gaussian control, Linear Quadratic tracker, State space two–stage optimizations, Finite horizon control, Infinite horizon control.*

1. Introduction

Objective of the paper is to solve the discrete–time linear state space optimum control problem. All the applied methods are based on two–stage optimizations with or without iteration. Design of control often involves some kind of limitation and optimization. The proposed methods limit the control signal through appropriate selection of the weighting factor of squared control signal in the performance index (PI). By this way the transient response and stability of the system can be influenced. The linear quadratic (LQ) PI based design in itself assures good robust capabilities that are proved in the professional literature. However, on an infinite horizon the control may become not satisfactory, but quality of it can be improved through suitable selection of weights. Constraints in different forms, too, may be given on the control signal. All the solutions in the paper use the same principles. For deterministic control, the optimization method "optimized trajectory tracking" (OTT, [1]) has been developed. OTT applies a sequence of optimizations on a section of two control stages. The corresponding method for stochastic systems is "optimized stochastic trajectory / output sequence tracking" (OSTT). The logic of two–stage optimizations is that the minimum of a PI on a finite horizon is not equal in general with the sum of one–stage minima, but can be computed through a sequence of two–stage optimizations. The consecutive two–stage optimizations overlap each other on one stage, the second stage of the preceding section and the first stage of the new section are common. The optimum linear state space solution of the optimum infinite horizon control problem (computation of the steady state optimum feedback gain) can be derived from optimum finite horizon control through limit value calculation of a part of the result of two–stage optimization. The methods shown in the paper can be extended for linear and nonlinear model predictive control and nonlinear stochastic tracking.

2. State Space Linear Optimum Control

2.1. State space linear optimum tracking

Consider a completely state controllable discrete - time system with constant parameters, given with the equations

$$\boldsymbol{x}(i+1)=\boldsymbol{A}\boldsymbol{x}(i)+\boldsymbol{B}\boldsymbol{u}(i)+\boldsymbol{w}(i), \quad (1)$$

$$\boldsymbol{y}(i)=\boldsymbol{C}\boldsymbol{x}(i)+\boldsymbol{n}(i). \quad (2)$$

In (1), (2) $\boldsymbol{x} \in \boldsymbol{R}^n$ denotes the state vector, $\boldsymbol{u} \in \boldsymbol{R}^m$ is the control vector, $\boldsymbol{y} \in \boldsymbol{R}^m$ is the output vector, $\boldsymbol{w}(i)$

and $\boldsymbol{n}(i)$ are white noise processes with zero mean and with the variances $E\{\boldsymbol{w}(i)\boldsymbol{w}^T(i)\}=\boldsymbol{R}_1$; $E\{\boldsymbol{n}(i)\boldsymbol{n}^T(i)\}=\boldsymbol{R}_2$ and $E\{\boldsymbol{w}(i)\boldsymbol{n}^T(i)\}=\boldsymbol{R}_{12}$, where E stands for expectation, and $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ are appropriate matrices. Assume that $E\{\boldsymbol{x}(0)\}$ is known, $\boldsymbol{u}(0)$ is estimated and the objective is tracking. Solution of the tracking problem may be based on fitting the estimated outputs to the references. If there are no constraints to take into consideration, optimum fitting may be achieved through making equal the next estimated output with the corresponding reference. It has been shown, however, that optimum fitting with respect to a PI, if there are constraints on the control signals is achieved through two–stage optimizations in general [2]. A series of two–stage optimizations and iteration give possibilities for computation of the time varying optimum feedback gain, associated with the concrete problem. An N – stage PI for tracking may be given as

$$F_{1,N} = E\left\{\sum_{i=1}^{N} \left(\boldsymbol{y}(\boldsymbol{x}(\mathrm{i})) - \boldsymbol{r}(i)\right)^T \boldsymbol{Q}\left(\boldsymbol{y}(\boldsymbol{x}(\mathrm{i})) - \boldsymbol{r}(\mathrm{i})\right) + \sum_{i=1}^{N} \boldsymbol{u}^T(i-1)\boldsymbol{R}\boldsymbol{u}(i-1)\right\}. \quad (3)$$

In (3) $\boldsymbol{Q}$ and $\boldsymbol{R}$ are appropriate weighting matrices, $\boldsymbol{r}(i)$ is the reference, assumed to be known for the whole horizon, N is the length of finite horizon in steps. $E\{\boldsymbol{y}(\boldsymbol{x}(N))\} \approx \boldsymbol{r}(N)$ may be a requirement for the desired end point of tracking. The optimization problem may be solved with OSTT. The two–stage PI can be given in the form

$$F_{i+1,i+2} = E\left\{\sum_{j=i+1}^{i+2} \left(\boldsymbol{y}(\boldsymbol{x}(\mathrm{j})) - \boldsymbol{r}(j)\right)^T \boldsymbol{Q}\left(\boldsymbol{y}(\boldsymbol{x}(\mathrm{j})) - \boldsymbol{r}(j)\right) + \sum_{j=\mathrm{i}+1}^{i+2} \boldsymbol{u}^T(j-1)\mathbf{R}\boldsymbol{u}(j-1)\right\}. \quad (4)$$

Solution of the tracking problem may be obtained through proof of the following theorem:

THEOREM :

The control vector that minimizes (4) can be given in the form

$$\boldsymbol{u}(\mathrm{i}) = -\boldsymbol{K}(i)\,\hat{\boldsymbol{x}}(i/i) + \boldsymbol{v}(i), \quad (5)$$

where

$\boldsymbol{K}(i)=\boldsymbol{K}(\boldsymbol{x}(i),\boldsymbol{x}(i+2))$ – time varying optimum feedback gain, computed from solution of the corresponding deterministic problem,

$\boldsymbol{v}(i)=\boldsymbol{v}(\boldsymbol{r}(i+1), \boldsymbol{r}(i+2))$ – command input, $\boldsymbol{r}(0)=0$.

PROOF:

The unknown control vector can be computed from

$$\frac{\partial F_{i+1,i+2}}{\partial \boldsymbol{u}(i)} = 0. \quad (6)$$

Since in case of linear processes with constant parameters and quadratic PI [3]

$$\min E\{\mathrm{PI}\} = E\{\min(\mathrm{PI})\}, \quad (7)$$

first minimum of (4) is computed without taking the expectation. From (6), taking into consideration that derivative of a quadratic function is linear,

$$\boldsymbol{u}(i)=\boldsymbol{f}\{\boldsymbol{u}(i+1),\boldsymbol{x}(i),\boldsymbol{r}(i+1),\boldsymbol{r}(i+2),\boldsymbol{w}(i),\boldsymbol{w}(i+1),\boldsymbol{n}(i+1),\boldsymbol{n}(i+2)\}. \quad (8)$$

To obtain $\boldsymbol{u}(i)$ in the form of (8), $\boldsymbol{x}(i+1)$ and $\boldsymbol{x}(i+2)$ are expressed with the state equations. Owing the property of (6), all terms in (8) are of first degree. (8) can be separated to parts which depend on $\boldsymbol{r}(i+1),\boldsymbol{r}(i+2),\boldsymbol{w}(i),\boldsymbol{w}(i+1),\boldsymbol{n}(i+1),\boldsymbol{n}(i+2)$ and which don't,

$$\boldsymbol{u}(i)=\boldsymbol{v}\{\boldsymbol{r}(i+1),\boldsymbol{r}(i+2),\boldsymbol{w}(i),\boldsymbol{w}(i+1),\boldsymbol{n}(i+1),\boldsymbol{n}(i+2)\}+\boldsymbol{f}_1\{\boldsymbol{u}(\mathrm{i}+1),\boldsymbol{x}(i)\}. \quad (9)$$

Taking into consideration that $\boldsymbol{u}(i)$, as the control signal on the first stage of a two-stage section is known, $\boldsymbol{u}(i+1)$ can be computed with (20). Since all terms of $\boldsymbol{f}_1$ are of first degree and $\boldsymbol{u}(\mathrm{i}+1)$ is linear function of $\boldsymbol{x}(i)$, $\boldsymbol{x}(i+2)$ and $\boldsymbol{u}(i)$, (9) may be written as

$$\boldsymbol{u}(i)=\boldsymbol{v}\{\boldsymbol{r}(i+1),\boldsymbol{r}(i+2),\boldsymbol{w}(i),\boldsymbol{w}(i+1),\boldsymbol{n}(i+1),\boldsymbol{n}(i+2)\}+\boldsymbol{f}_2\{\boldsymbol{x}(i),\boldsymbol{x}(\mathrm{i}+2)\}. \quad (10)$$

$\boldsymbol{u}(i+1)$ cannot be expressed with $\boldsymbol{x}(i)$ and $\boldsymbol{x}(i+2)$ in general with matrix operations, since the dimensions of the vectors are different in general. However, elements of $\boldsymbol{u}(i+1)$ may be expressed from the state equations with elements of $\boldsymbol{x}(i)$ and $\boldsymbol{x}(i+2)$ through some considerations. $\boldsymbol{f}_2$ is also a polynomial of first degree and can be expressed as

$$\boldsymbol{f}_2=-\boldsymbol{K}_1\boldsymbol{x}(i)-\boldsymbol{K}_2\boldsymbol{x}(i+2), \quad (11)$$

where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are appropriate constant matrices. If (11) is written in the form of the linear state space feedback control law [4],

$$\boldsymbol{f}_2=-\boldsymbol{K}(i)\boldsymbol{x}(i). \tag{12}$$

From (11) and (12),

$$\boldsymbol{K}(i)\boldsymbol{x}(i)=\boldsymbol{K}_1\boldsymbol{x}(i)+\boldsymbol{K}_2\boldsymbol{x}(i+2). \tag{13}$$

For computation of the time varying optimum feedback gain, $\boldsymbol{K}_2\boldsymbol{x}(i+2)$ may be transformed to $\boldsymbol{K}_3\boldsymbol{x}(i)$:

$$\boldsymbol{K}_2\boldsymbol{x}(i+2)=\boldsymbol{K}_3\{\boldsymbol{x}(i),\boldsymbol{x}(i+2)\}\boldsymbol{x}(i), \tag{14}$$

$$K_3^{p,q}=\frac{k_2^{p,q}x_q(i+2)}{x_q(i)}, 1\le p\le m, 1\le q\le n, \tag{15}$$

where p,q are serial numbers of elements of matrices $\boldsymbol{K}_2$, $\boldsymbol{K}_3$. From (13) and (15)

$$\boldsymbol{K}(i)=\boldsymbol{K}_1+\boldsymbol{K}_3\{\boldsymbol{x}(i),\boldsymbol{x}(i+2)\}. \tag{16}$$

Solution of the control problem from (10), (11), (13) and (16) is

$$E\{\boldsymbol{u}(i)\}=E\,\{\boldsymbol{v}(\boldsymbol{r}(i+1),\boldsymbol{r}(i+2),\boldsymbol{w}(i),\boldsymbol{w}(i+1),\boldsymbol{n}(i+1),\boldsymbol{n}(i+2))-\boldsymbol{K}(i)\boldsymbol{x}(i)\} \tag{17}$$

$$=\boldsymbol{v}\big(\boldsymbol{r}(i+1),\boldsymbol{r}(i+2)\big)-\boldsymbol{K}(i)\,\hat{x}\,(i).$$

The above result can be obtained taking into consideration that at time point t_i $\boldsymbol{x}(i+2)$ can be estimated only through prediction, and computation of the time varying optimum feedback gain gives the same result as in the deterministic case, if $\hat{x}\,(i)=\boldsymbol{x}(i)$. Moreover, since the gains have to be computed in advance on the whole finite horizon, all the future disturbances have to be taken into account with their expectations, resulting in a deterministic model. Consequently, gains of the corresponding deterministic model can be used in the stochastic case, too. With alternative sequence of thoughts, the problem can be solved for the deterministic case, replacing the states with their concrete values in the gains, and the certainty equivalence principle can be used for the solution in the stochastic case. Applicability of the certainty equivalence principle can be proved for linear constant parameter systems with quadratic PI and white noise disturbances [3]. In (17) [5]

$$\hat{x}\,(\mathrm{i})\;=E\{\boldsymbol{x}(i)/\boldsymbol{Y}(i)\}. \tag{18}$$

In (18) $\hat{x}\,(i)$ is the Kalman – filter estimate and $\boldsymbol{Y}$(i) is the measurement vector.

In the deterministic case the states can be computed exactly, and the control law from (17) with $\hat{x}=\boldsymbol{x}$ substitution is

$$\boldsymbol{u}(i)=\boldsymbol{v}(\boldsymbol{r}(i+1),\boldsymbol{r}(i+2))-\boldsymbol{K}(i)\boldsymbol{x}(i). \tag{19}$$

(17) and (19) are the stochastic and deterministic feedback solutions of the problem. The feedforward solution for stochastic systems can be obtained from (8) through expressing $\boldsymbol{u}(i+1)$ from it and taking the expectation

$$E\{\boldsymbol{u}(i+1)\}=E\{\boldsymbol{f}_3\big(\boldsymbol{u}(i),\boldsymbol{x}(i),\boldsymbol{r}(i+1),\boldsymbol{r}(i+2),\boldsymbol{w}(i),\boldsymbol{w}(i+1),\boldsymbol{n}(i+1).\boldsymbol{n}(i+2)\big)\} \tag{20}$$

$$=f_3\Big(\boldsymbol{u}(i),\,\hat{x}\,(i),\boldsymbol{r}(i+1),\boldsymbol{r}(i+2)\Big),$$ and for deterministic systems

$$\boldsymbol{u}(i+1)=\boldsymbol{f}_3\big(\boldsymbol{u}(i).\boldsymbol{x}(i),\boldsymbol{r}(i+1).\boldsymbol{r}(i+2)\big). \tag{21}$$

Even if there is no unique feedback solution, a unique feedforward solution may be obtained, if the system (1), (2) is reachable. With the above algorithms both fixed end point ($\boldsymbol{y}(N)\approx\mathbf{r}(N)$ for an $\boldsymbol{N}$-stage control problem) and free end point ($\boldsymbol{y}(N)\neq\boldsymbol{r}(N)$) problems can be solved. At the beginning of computations, an estimation is made for the $\boldsymbol{u}(0)$ control vector on the first stage, e.g. from the

$$\boldsymbol{y}(1)\approx\boldsymbol{r}(1) \tag{22}$$

approximation. If feedforward computation is used, $\boldsymbol{u}(1)$ and $\boldsymbol{y}(2)$ can be obtained in knowledge of $\boldsymbol{x}(0)$, $\boldsymbol{y}(1)$ and $\boldsymbol{u}(0)$. In the next step, optimization is made on the section [$\boldsymbol{y}(1)$, $\boldsymbol{y}(3)$], from which $\boldsymbol{u}(2)$ and $\boldsymbol{y}(3)$ can be computed. Through this procedure $\boldsymbol{y}(N)\neq\boldsymbol{r}(N)$ is obtained from N-1 section optimization. If the references are known in advance and $\boldsymbol{y}(N)\approx\boldsymbol{r}(N)$ is a requirement, the optimum $\boldsymbol{u}(0)$ control vector can be obtained through iterations (regula falsi, Newton iteration method etc.).The obtained output sequence, if the optimization problem has only one unique solution, is optimum, since in the [$\boldsymbol{y}(i-1)$, $\boldsymbol{y}(i+1)$] section $\boldsymbol{y}(i+1)$ is optimum for a given $\boldsymbol{y}(i-1)$, $\boldsymbol{u}(i-1)$. However, in consequence of the iterative character of the solution, the final result is practically suboptimal (although deviation at the end point may be very small). (5) shows that the tracker is noncausal, because future values of the states are needed to compute the time varying optimum feedback gain. This fact necessitates to solve the control problem (to estimate the states on the whole horizon) in advance. The $\boldsymbol{K}$ gain in (5) is plant and PI dependent and may be called "tracker gain". For infinite horizon control, the steady state optimum feedback gain has to be computed. In this case, it is enough to optimize on a section of two control stages, without computing a whole trajectory in advance and without iteration. The steady state gain is obtained as limit value and the limit doesn't depend on values of concrete final states [6]:

$$\boldsymbol{K}(\infty) = \lim_{i\to\infty} \boldsymbol{K}\big(\boldsymbol{x}(i), \boldsymbol{x}(i+2)\big). \tag{23}$$

Control with steady state feedback has the advantage of simplicity and good stability properties.

2.2. Linear Quadratic Gaussian control

Consider the (1), (2) system. For Linear Quadratic Gaussian (LQG) control the PI may be given in the form [4] of

$$F_{1,N} = E\{\textstyle\sum_0^{N-1} \quad [\boldsymbol{x}^T(i)\boldsymbol{Q}_1 x(i) + 2x^T(i)\boldsymbol{Q}_{12}\boldsymbol{u}(i) + \boldsymbol{u}^T(i)\boldsymbol{Q}_2 u(i)] + \boldsymbol{x}^T(N)\boldsymbol{Q}_0\boldsymbol{x}(N)\}. \quad 24)$$

In (24) $\boldsymbol{Q}_1$, $\boldsymbol{Q}_{12}$, $\boldsymbol{Q}_2$, $\boldsymbol{Q}_0$ are approximate weighting matrices. The (24) PI is different from that of (3), but in both PIs all the terms are quadratic and in the (6) derivative all the terms are linear. Consequently, the sequence of thoughts of the above PROOF can be applied. Since $\boldsymbol{r}(i)=0$, $i=0,1,2,\ldots N$ for LQG control, the control law from (17) becomes

$$\boldsymbol{u}(i) = -\boldsymbol{K}(i)\,\hat{\boldsymbol{x}}\,(i). \tag{25}$$

However, $\boldsymbol{K}(i)$ in (25) is different from $\boldsymbol{K}(i)$ in (17), which is the tracker gain. Conventionally, the $\boldsymbol{K}(i)$ gain comes from solution of the Riccati equation. This paper gives an alternative and simpler way to compute the gain matrix for LQG control. If there is only one unique solution of the control problem, the gain in (25) should be approximately equal to the gain of LQG control computed through solution of the Riccati equation for the same system, PI, and for approximately same conditions. The difference comes from the iterative character of the solution in the paper. OTT and OSTT can be used beyond the sphere of LQ control, even for control of nonlinear plants [2], too.

3. Example

This example shows a SISO deterministic state space optimum control design. The state and output equations of the plant are

$$x(i+1)=ax(i)+bu(i), \tag{26}$$

$$y(i)=x(i). \tag{27}$$

In (26), a and b are parameters. The two-stage PI is given as

$$F_{i+1.i+2} = \textstyle\sum_{j=i+1}^{i+2} \quad \left\{\big(r(j)-x(j)\big)^2 + \lambda u^2(j-1)\right\}. \tag{28}$$

In (28) λ is the control weight. With (6), the unknown control signal is

$$u(i+1)=r(i+1)/(ab)+r(i+2)/b-x(i)(1+a^2)/b-u(i)\{b^2(1+a^2)+\lambda\}/(ab^2). \tag{29}$$

With elimination of $u(i+1)$ through the state equations

$$u(i) = \frac{ab}{b^2+\lambda}\left\{\frac{r(i+1)}{a} + r(i+2)\right\} - \frac{ab}{b^2+\lambda}\left\{1 + \frac{x(i+2)}{x(i)}\right\}x(i) = v(i) - K(i)x(i). \tag{30}$$

(29), (30) can be used for computation on a finite horizon. The $K(i)$ time varying optimum feedback gain in (30) needs to be computed in advance for all stages of the horizon. For infinite horizon control and unit step reference, the steady state tracker gain, taking into consideration that

$$\lim_{i\to\infty} \frac{x(i+2)}{x(i)} = 1, \tag{31}$$

is

$$K_{tr}(\infty) = \lim_{i\to\infty} K(i) = \frac{2ab}{b^2+\lambda}. \tag{32}$$

The steady state optimum feedback gain for LQG control may be computed by (23) through application of the l'Hospital rule. $x(i+2)$ can be expressed in function of $x(i)$, $u(i)$ and $u(i+1)$:

$$x(i+2)=a^2x(i)+abu(i)+bu(i+1). \tag{33}$$

With (33) and (30)

$$\lim_{i\to 0}\frac{\partial x(i+2)}{\partial x(i)}=\frac{a^2\lambda}{b^2+\lambda},\tag{34}$$

$$K_{l\,q\,g}(\infty)=\frac{ab}{b^2+\lambda}\left(1+\frac{a^2\lambda}{b^2+\lambda}\right).\tag{35}$$

For the closed loop with (26), (30) and (23)

$$x(i+1)=\frac{ab^2}{b^2+\lambda}\left\{\frac{r(i+1)}{a}+r(i+2)\right\}+\left(a-b\,K(\infty)\right)x(i).\tag{36}$$

If λ=0.

$$K_{t\,r}(\infty)=\frac{2a}{b},\ K_{l\,q\,g}(\infty)=\frac{a}{b}.\tag{37}$$

For the tracker if λ=0 with (36), (37)

$$x(i+1)=r(i+1)+ar(i+2)\text{-}ax(i).\tag{38}$$

In steady state ($r(i)=r(i+1)=r(i+2)\ldots,x(i)=x(i+1)=x(i+2)\ldots,$)

$$x(i)+ax(i)=r(i)+ar(i).\tag{39}$$

From (39)

$$x(i)=r(i).\tag{40}$$

For LQG if λ= 0 with (36), (37)

$$x(i+1)=\left(a-bK_{l\,q\,g}(\infty)\right)x(i)=0.\tag{41}$$

If the reversed gains were applied, for LQG with the tracker gain

$$x(i+1)=\text{-}ax(i),\tag{42}$$

the final state is approached through oscillations. For tracker with the LQG gain

$$x(i+1)=r(i+1)+ar(i+2),\tag{43}$$

the steady state error is increased.

4. Conclusion

The paper demonstrates a new approach to optimum solution of the LQG control and LQ tracking problems in the state space. The principle of solution is finite horizon control design through a sequence of two–stage optimizations and iteration, applying formerly worked out optimization methods. Infinite horizon control may be achieved from finite horizon one through limit value calculation. The proposed methods make possible simple design of robust control systems satisfying given specifications. The solution of the problem gives a new method to computation of the time varying optimum feedback gain, both for LQG control and LQ tracking. An example shows the applicability of ideas presented in the paper.

References

[1] E. Nagy, Optimum Control of Sampling Systems through Optimized Trajectory Tracking, Conference Proceedings, American Control Conference, Pp. 3897-3901, (1997).

[2] E. Nagy, Control of Distributed Parameter Systems: a Practical Approach, Conference Proceedings, Sixth International Conference on Mathematical Problems in Engineering & Aerospace Sciences, Pp. 527-539, (2006).

[3] R. Isermann, *Digital Control Systems*, Berlin: Springer – Verlag, 1981, p. 275.

[4] K.J. Aström and B. Wittenmark, *Compute–Controlled Systems: Theory and Design*, Englewood Cliffs: Prentice – Hall, Inc.1984, Pp.264, 257.

[5] J.V. Candy, *Signal Processing,* New York: McGraw – Hill Book Company, 1987, p.72.

[6] E. Nagy, Unified Approach to Optimization, Control and Estimation of Discrete Time Lumped Parameter Systems, Conference Proceedings, 5th International Conference on Mathematical Problems in Engineering and Aerospace Sciences, Pp. 503 – 514, (2004).

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