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Lie-Theoretic Generating Relations of Two Variable Chebyshev Polynomials

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Abstract: *In this paper, we appeal to the Lie theoretic concepts and derive generating function relations involving 2-variable Chebyshev polynomials of the second kind U_n (x,y). For this class of polynomials, we give suitable interpretations to the index n of the polynomials U_n (x,y) with the help of Weisner's method and the approach requires the construction of 3-dimensional Lie algebra isomorphic to sl(2), the special linear algebra.*

1. Introduction

Many different approaches were appeared to understanding the special functions of Mathematical Physics. These functions play an important role in mathematical physics and appear in it as solutions to various differential equations describing many different kinds of its systems. All of them share common features, considering the variations in the forms of special functions that exist. Special functions satisfy various differential and recursion relations, have generating function relations and are solutions to certain second-order differential equations.

Marius Sophus Lie (1842-1899) [1] attempts to use the power of the tool called group theory to solve, or at least simplify ordinary differential equations. Earlier in the nineteenth century, Evariste Galois (1811-1832) had used group theory to solve algebraic (polynomial) equations that were quadratic, cubic, and quartic.

Based on the comparison, Lie initiated his work. If finite groups were needed to determine the solvability of finite-degree polynomial equations, then the treatment of ordinary and partial differential equations would probably include infinite groups.

The Chebyshev polynomials are one of the most useful polynomials so it's hard to avoid Chebyshev polynomials. In just about every mathematics branch, they appear, including spectral methods for partial differential equations, geometry, combinatorics, number theory, the theory of approximation, numerical analysis, statistics, integral and differential equations[2, 3, 4, 5].

There are several distinct families of polynomials that go by the name of Chebyshev polynomials[6, 7, 8].

We focus our attention to the Chebyshev polynomials of the second kind which are defined by [9].

$$U_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k (2x)^{n-2k}, \tag{1}$$

A natural generalization of the classical polynomials in one variable[10] is the Chebyshev polynomials in several variables. They are used in numerous fields of mathematics (e.g. discrete analysis, approximation theory[11], linear algebra[12],[13], representation theory[14]-[16]) and physics[17]-[21], respectively).

In this paper, we appeal to the Lie theoretic concepts and derive generating function relations involving 2-variable Chebyshev polynomials of the second kind $U_n(x,y)$. For this class of polynomials, we give suitable interpretations to the index n of the polynomials $U_n(x,y)$ with the help of Weisner's method and the approach requires the construction of 3-dimensional Lie algebra isomorphic to $sl(2)$, the special linear algebra.

We begin by introducing the 2-variable Chebyshev polynomials of the second kind

$$U_n(x,y) = \sum_{k=0}^{[n/2]} n-k_{\,k}\; (-y)^k (2x)^{n-2k}, \tag{2}$$

which related to the 1-variable Chebyshev polynomials of the second kind by the relation

$$U_n(x,1) = U_n(x). \tag{3}$$

Consider the following differential equation which satisfied by the 2-variable Chebyshev polynomials of the second kind $w = U_n(x,y)$:

$$(x^2-y)\frac{d^2w}{dx^2} + 3x\frac{dw}{dx} - n(n+2)w = 0. \tag{4}$$

2. GROUP-THEORETIC DISCUSSION

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $z\frac{\partial}{\partial z}$ and w by $u(x,y,z)$ in (1.4), we get the following partial differential equation:

$$(x^2-y)\frac{\partial^2 u}{\partial x^2} - z^2\frac{\partial^2 u}{\partial z^2} + 3x\frac{\partial u}{\partial x} - 3z\frac{\partial u}{\partial z} = 0. \tag{1}$$

Thus $u(x,y,z) = z^n U_n(x,y)$ is a solution of (2.1), since $U_n(x,y)$ is a solution of (1.4).

We now seek linearly independent differential operators $J^3, J^-$and J^+ such that

$$\begin{cases} J^3(z^n U_n(x,y)) = a_n z^n U_n(x,y) \\ J^-(z^n U_n(x,y)) = b_n z^{n-1} U_{n-1}(x,y) \\ J^+(z^n U_n(x,y)) = b_n z^{n+1} U_{n+1}(x,y) \end{cases}$$

(2)

where a_n, b_n and c_n are expressions in n which are independent of x, y and z.

This necessitates the bringing into use of the following recurrence relations.

The 2-variable Chebyshev polynomials of the second kind $U_n(x,y)$ satisfies the following recurrence relations

$$(ii)(x^2-y)\frac{\partial}{\partial x}U_n(x,y) = (n+1)U_{n+1}(x,y) - (n+2)xU_n(x,y);$$

$$(iii)(x^2-y)\frac{\partial}{\partial x}U_n(x,y) = nxU_n(x,y) - (n+1)yU_{n-1}(x,y);$$

By the aid of the above recurrence relations and (2.2), we get the following differential operators

$$\begin{cases} J^3 = z\frac{\partial}{\partial z} + 1 \\ J^- = z^{-1}y^{-1}[(x^2-y)\frac{\partial}{\partial x} - xzy\frac{\partial}{\partial z}] \\ J^+ = z[(x^2-y)\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z} + 2x] \end{cases}$$

(3)

such that

$$J^3(z^n U_n(x,y)) = (n+1)z^n U_n(x,y)$$

$$J^-(z^n U_n(x,y)) = -(n+1)z^{n-1} U_{n-1}(x,y)$$

$$J^+(z^n U_n(x,y)) = (n+1)z^{n+1} U_{n+1}(x,y)$$

Now we have the following commutator relations:

$$[J^3, J^{\pm}] = \pm J^{\pm}, \qquad [J^+, J^-] = 2J^3, \tag{4}$$

where $[A,B]u = (AB - BA)u$.

The above commutator relations (2.4) show the set of operators $J^3, J^-, J^+, 1$ where 1 stands for identity operator, generates a Lie algebra which is isomorphic to $sl(2)$.

Now we can write the differential operator as follow

$$\frac{(x^2-y)}{y} L \equiv J^+J^- + J^3(J^3 - 1) \tag{5}$$

it follows that $f(x,y)L$ commutes with each operator

$$[J^3, f(x,y)L] = [J^-, f(x,y)L] = [J^+, f(x,y)L] = 0 \tag{6}$$

where $f(x,y) = \frac{(x^2-y)}{y}$

Extended forms of the groups generated by J-operators

To find the extended form of the group generated by J^3, we must solve the following differential equations

$$\frac{\partial z(a')}{\partial a'} = z(a') \tag{7}$$

and

$$\frac{\partial v(a')}{\partial a'} = v(a') \tag{8}$$

where $z(0) = z$, and $v(0) = 1$.

From (2.7), we get

$$\int \frac{\partial z(a')}{\partial a'} \frac{1}{z(a')} da' = \int da'$$

$$\log z(a') = a' + k$$

We put $a' = 0$ to find the constant k, so we get

$$\log z = k$$

Then

$$\log z(a') = a' + \log z$$

$$z(a') = z e^{a'}$$

and from (2.8), we get

$$\int \frac{\partial v(a')}{\partial a'} \frac{1}{v(a')} da' = \int da'$$

$$\log v(a') = a' + k$$

We put $a' = 0$ to find the constant k, so we get

$$k = 0$$

Then

$$\log v(a') = a'$$
$$v(a') = e^{a'}$$

Thus

$$e^{a' J^3} u(x,y,z) = e^{a'} u(x,y,ze^{a'}) \tag{9}$$

Similarly we get

$$e^{b' J^-} u(x,y,z) = u\left(\frac{\alpha\sqrt{y}}{\sqrt{\alpha^2 - z^2\beta}}, y, \frac{\sqrt{\alpha^2 - z^2\beta}}{\sqrt{y}}\right) \tag{10}$$

and

$$e^{c' J^+} u(x,y,z) = \left(\frac{y}{\gamma^2 - \beta}\right) u\left(\frac{\gamma\sqrt{y}}{\sqrt{\gamma^2 - \beta}}, y, \frac{z\sqrt{y}}{\sqrt{\gamma^2 - \beta}}\right) \tag{11}$$

where $\alpha = (xz - b')$, $\beta = (x^2 - y)$ and $\gamma = x - c' yz$

Therefore

$$e^{c' J^+} e^{b' J^-} u(x,y,z) =$$
$$(\gamma^2 - \beta)^{-\left(1+\frac{n}{2}\right)} y^{\left(1+\frac{n}{2}\right)} (y^2 z^2 - 2b' yz\gamma + b'^2(\gamma^2 - \beta))^{\frac{n}{2}} \tag{12}$$

$$\times U_n\left(\frac{\sqrt{y}(yz\gamma - b'(\gamma^2 - \beta))}{\sqrt{(\gamma^2 - \beta)(y^2 z^2 - 2b' yz\gamma + b'^2(\gamma^2 - \beta))}}, y\right)$$

3. Generating Function Relations

We will obtain generating function relations from the operator J^3 by considering the following two cases of the transformed function $\exp(c' J^+)\exp(b' J^-)U_n(x,y)$

From (2.1) $u(x,y,z) = z^n U_n(x,y)$ is a solution of the system

$$\begin{cases} Lu = 0, \\ (A - (n+1))u = 0. \end{cases} \tag{1}$$

Since $f(x,y)L$ commutes with the operators, we have

$$S(f(x,y)L)(z^n U_n(x,y)) = (f(x,y)L)S(z^n U_n(x,y)) = 0$$

where

$$S = e^{c' J^+} e^{b' J^-}$$

Therefore, the transformation $S(z^n U_n(x,y))$ is also annulled by $f(x,y)L$.

So, we consider the following special cases:

Case I : Let $b' = 1$, $c' = 0$, then (2.12) reduces to

$$e^{J^-}[z^n U_n(x,y)] = \left(z^2 - \frac{2zx}{y} + \frac{1}{y}\right)^{\frac{n}{2}} \qquad U_n\left(\frac{zx - 1}{\sqrt{z^2 - \frac{2zx}{y} + \frac{1}{y}}}, y\right) \tag{2}$$

Now expanding this function, we get

$$e^{J^-}[z^n U_n(x,y)] = \sum_{m=0}^{\infty} \frac{(J^-)^m}{m!} [z^n U_n(x,y)]$$

$$= \sum_{m=0}^{\infty} \frac{(J^-)^{m-1}}{m!}[-(n+1)z^{n-1}U_{n-1}(x,y)]$$

$$= \sum_{m=0}^{\infty} \frac{(J^-)^{m-m}}{m!}[(-(n+1))_m z^{n-m}U_{n-m}(x,y)]$$

Thus

$$e^{J^-}[z^n U_n(x,y)] = \sum_{m=0}^{\infty} \frac{(-(n+1))_m}{m!}[z^{n-m}U_{n-m}(x,y)] \quad (3)$$

Equating the two equations (3.2) and (3.3), we get

$$(z^2 - \frac{2zx}{y} + \frac{1}{y})^{\frac{n}{2}} \qquad U_n(\frac{zx-1}{\sqrt{z^2-\frac{2zx}{y}+\frac{1}{y}}}, y) \quad (4)$$

$$= \sum_{m=0}^{\infty} \frac{(-(n+1))_m}{m!}[z^{n-m}U_{n-m}(x,y)]$$

$$(1 - \frac{2x}{yz} + \frac{1}{yz^2})^{\frac{n}{2}} \qquad U_n(\frac{x-\frac{1}{z}}{\sqrt{1-\frac{2x}{yz}+\frac{1}{yz^2}}}, y) \quad (5)$$

$$= \sum_{m=0}^{\infty} \frac{(-(n+1))_m}{m!}[z^{-m}U_{n-m}(x,y)]$$

Let $t = \frac{1}{z}$

$$(1 - \frac{2xt}{y} + \frac{t^2}{y})^{\frac{n}{2}} \qquad U_n(\frac{x-t}{\sqrt{1-\frac{2tx}{y}+\frac{t^2}{y}}}, y) \quad (6)$$

$$= \sum_{m=0}^{\infty} \frac{(-(n+1))_m}{m!}U_{n-m}(x,y)t^m$$

Case II : Let $b' = 0,\ c' = 1$, then (2.12) reduces to

$$e^{J^+}[z^n U_n(x,y)] = (yz^2 - 2xz + 1)^{-(1+\frac{n}{2})} z^n U_n(\frac{x-yz}{\sqrt{yz^2-2xz+1}}, y) \quad (7)$$

Now expanding this function, we get

$$e^{J^+}[z^n U_n(x,y)] = \sum_{k=0}^{\infty} \frac{(J^+)^k}{k!}[z^n U_n(x,y)]$$

$$= \sum_{k=0}^{\infty} \frac{(J^+)^{k-1}}{k!}[(n+1)z^{n+1}U_{n+1}(x,y)]$$

$$= \sum_{k=0}^{\infty} \frac{(J^+)^{k-k}}{k!}[(n+1)_k z^{n+k}U_{n+k}(x,y)]$$

Thus

$$e^{J^+}[z^n U_n(x,y)] = \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!}[z^{n+k}U_{n+k}(x,y)] \quad (8)$$

Equating the two equations (3.7) and (3.8), we get

$$(yz^2 - 2xz +$$

$$1)^{-(1+\frac{n}{2})}U_n(\frac{x-yz}{\sqrt{yz^2-2xz+1}},y) \quad (9)$$

$$= \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} U_{n+k}(x,y)z^k$$

Finally, we want to point out that the interchanging of the order of the operators in (2.12) will give different results.

4. Particular Cases

We can obtain the following generating function relations for the basic Chebyshev polynomials.

I: Taking $y = 1$ in (3.6), we get

$$(1-2xt+t^2)^{\frac{n}{2}} \qquad U_n(\frac{x-t}{\sqrt{1-2tx+t^2}}) = \sum_{m=0}^{\infty} \frac{(-(n+1))_m}{m!} U_{n-m}(x)t^m$$

II: Taking $y = 1$ in (3.9), we get

$$(z^2-2xz+1)^{-(1+\frac{n}{2})}U_n(\frac{x-z}{\sqrt{z^2-2xz+1}}) = \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} U_{n+k}(x)z^k$$

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