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The Inequalities of Positive Semi-Definite Block Matrix with Partial Order Relations

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Abstract: *In 2019, Zübeyde Ulukök obtained an important theorem in reference [1]: When* $H=\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ *is a positive semi-definite matrix, then* $H^r \le 3[\lambda_1(A)+\lambda_1(B)]^{r-1}\begin{bmatrix} A & \\ & B \end{bmatrix}$ *,where* A, B *are n-order square matrices. In this paper, we firstly do the same thing for a* 3×3 *positive semi-definite block matrix, and give a generalization of the above theorem. Next, we further generalize the case of* $k\times k$ *positive semi-definite block matrix, and discuss the partial ordering relationship between the sum of matrices on quasi-diagonal lines and block matrices at other locations. Thus, we gave a new eigenvalue inequality.*

Keywords: *Positive Semi-Definite Matrix Hermitian Matrix Partial Order Relation*

1. Introduction and Preliminaries

Inequalities of positive semi-definite block matrices have been widely used in matrix theory. In recent years, inequalities about block matrices have become a hot topic of research. At the same time, some very good results have been obtained, such as references [1, 3, 4, 5]. In this paper, we mainly discussed some positive semi-definite block matrices and obtained some matrix inequalities.

As we all know, the positive semi-definite block matrices have very good properties. Their eigenvalues are real numbers, so they can always be arranged in ascending order and we recorded then as $\lambda_n(A)\le\cdots\le\lambda_2(A)\le\lambda_1(A)$. In this paper, we use symbol $\lambda_1(A)$ to represent the largest eigenvalue of a positive semi-definite matrix A. And use symbol $A\le B$ to represent $B-A$ be a positive semi-definite matrix, obviously, "$\le$" is a partial order relation. In particular, $A\ge 0$ denotes that matrix A is positive semi-definite. In addition, we call U a unitary matrix if it satisfies $U^*U=I$, we call A a Hermitian matrix if it satisfies $A^*=A$. Last, the $A\oplus B$ denotes the direct sum of A and B, the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$; and 0 represents a zero block matrix.

2. Main Results

Let's start with the following lemmas

Lemma1.1 Let $A\in M_{m,n}$ with $m\ge n$, then $\lambda(AA^*)=\lambda(A^*A\oplus 0)$ with $0\in M_{m-n}$.

Lemma1.2 Let $A\in M_n$ be positive semi-definite matrix. Then $\lambda_n(A)I\le A\le\lambda_1(A)I$, where I denotes the identity matrix in M_n.

Lemma1.3 If $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ is a positive semi-definite matrix, then $A \geq B$.

Proof Just consider the equation $A-B=\frac{1}{2}\begin{pmatrix} I & -I \end{pmatrix}\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} I \\ -I \end{pmatrix} \geq 0$ and we can get the conclusion.

Lemma1.4 If $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is a positive semi-definite matrix, A, C are square matrices of the same order, then we have $A+C \geq B+B^*$ and $A+C \geq -(B+B^*)$.

Proof Becase $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is a positive semi-definite matrix, So, for any unitary matrix U,we have $U^*\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}U \geq 0$, thus there is $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}+U^*\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}U \geq 0$. In particular, take $U=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$,then we have $\begin{pmatrix} A+C & B+B^* \\ B+B^* & A+C \end{pmatrix} \geq 0$.From lemma 1.3 we can get that

$A+C \geq B+B^*$. And we can draw another conclusion by replacing the original B with $-B$.

Lemma1.5 If $H \in M_n$ is a positive semi-definite matrix, then for any $n \times k$ unitary matrix U satisfying $U^*U = I_k$ and for each $i=1,\ldots,k$, we have $\lambda_{i+m-k}(H) \leq \lambda_i(U^*HU) \leq \lambda_i(H)$.

Theorem 1 $H=\begin{bmatrix} A & D & E \\ D^* & B & F \\ E^* & F^* & C \end{bmatrix}$ is a positive semi-definite matrix, where A, B, C are all n-order square matrices,then

$$H^r \leq 3[\lambda_1(A)+\lambda_1(B)+\lambda_1(C)]^{r-1}\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} \quad \text{for } r \geq 1.$$

ProofBecause H is a positive semi-definite matrix, so there exists an invertible matrix P such that $H = P^*P$. We divide P into blocks: $P=[X,Y,Z]$, where $X, Y, Z \in M_{3n\times n}$, and we can know that $X^*X = A,\ \ Y^*Y = B,\ \ Z^*Z = C.$ From Lemma 1.1, we can get the following results:

$$\lambda_1(XX^*) = \lambda_1(X^*X \oplus 0) = \lambda_1(A),\ \lambda_1(YY^*) = \lambda_1(Y^*Y \oplus 0) = \lambda_1(B),$$

$$\lambda_1(ZZ^*) = \lambda_1(Z^*Z \oplus 0) = \lambda_1(C).$$

Noting that the following equation holds: $H^r = (P^*P)^r = P^*(PP^*)^{r-1}P$,and we denote

$T=(PP^*)^{r-1}$. So, for H^r, there is the following decompositions :

$$H^r=P^*TP=\begin{bmatrix} X^*TX & X^*TY & X^*TZ \\ Y^*TX & Y^*TY & Y^*TZ \\ Z^*TX & Z^*TY & Z^*TZ \end{bmatrix}=\begin{bmatrix} X & & \\ & Y & \\ & & Z \end{bmatrix}^*\begin{bmatrix} T & T & T \\ T & T & T \\ T & T & T \end{bmatrix}\begin{bmatrix} X & & \\ & Y & \\ & & Z \end{bmatrix}$$

We notice that $\begin{bmatrix} T & T & T \\ T & T & T \\ T & T & T \end{bmatrix}=\begin{bmatrix} T^{\frac{1}{2}} & 0 & 0 \\ T^{\frac{1}{2}} & 0 & 0 \\ T^{\frac{1}{2}} & 0 & 0 \end{bmatrix}\begin{bmatrix} T^{\frac{1}{2}} & 0 & 0 \\ T^{\frac{1}{2}} & 0 & 0 \\ T^{\frac{1}{2}} & 0 & 0 \end{bmatrix}^*$, and $\lambda\begin{bmatrix} T & T & T \\ T & T & T \\ T & T & T \end{bmatrix}=3\lambda(T)$.

So, from lemma 1.2. We can get that:

$$H^r\le 3\lambda_1(T)\begin{bmatrix} X & & \\ & Y & \\ & & Z \end{bmatrix}^*\cdot\begin{bmatrix} X & & \\ & Y & \\ & & Z \end{bmatrix}=3\lambda_1(T)\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix}=3\lambda_1(PP^*)^{r-1}\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix}$$

$$=3\lambda_1^{r-1}(XX^*+YY^*+ZZ^*)\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix}\le 3[\lambda_1(XX^*)+\lambda_1(YY^*)+\lambda_1(ZZ^*)]^{r-1}\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix}$$

$$=3[\lambda_1(A)+\lambda_1(B)+\lambda_1(C)]^{r-1}\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix}$$. So, the theorem is proved.

On the basis of this conclusion, we can easily get the results of Zübeyde Ulukök in reference[1]:

Corollary 1 $H=\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is a positive semi-definite matrix, where A , B are n-order square matrices,then

$$H^r\le 3[\lambda_1(A)+\lambda_1(B)]^{r-1}\begin{bmatrix} A & \\ & B \end{bmatrix} \quad \text{for } r\ge 1.$$

Next, we will discuss the case of $k\times k$ positive semi-definite block matrices and explore the relationship between quasi-diagonal matrices and other block matrices at other locations.

Theorem 2 $H=\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{12}^* & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1k}^* & M_{2k}^* & \cdots & M_{kk} \end{bmatrix}$ is a positive semi-definite matrix, and $M_{11}, \ldots, M_{kk}$ are all n-order square matrices, if it satisfied M_{ij} are all Hermitian matrices, for

($1\le i\le j\le n$), then

$$\sum_{i=1}^{k} M_{ii}\ge\frac{2}{k-1}\sum_{1\le i<j\le n} M_{ij}$$

Proof First of all, notice the following facts: if $U_1,\ldots,\ U_k$ are all unitary matrices, A is a positive semi-definite matrix, then $U_1^*AU_1+\ldots U_k^*AU_k$ is still a positive semi-definite matrix.

After calculation, we can get that $\sum_{i=1}^{k-1}\sum_{l=i+1}^{k}\begin{pmatrix} M_{ii} & M_{il} \\ M_{il} & M_{ll} \end{pmatrix} \geq 0$. From lemma 1.4 we can get that :

$$\sum_{i=1}^{k-1}\sum_{l=i+1}^{k}(M_{ii}+M_{ll}) \geq \sum_{i=1}^{k-1}\sum_{l=i+1}^{k}(M_{il}+M_{il}) = 2\sum_{1\leq i<j\leq n} M_{ij},$$

So, there is $\sum_{i=1}^{k} M_{ii} \geq \frac{2}{k-1}\sum_{1\leq i<j\leq n} M_{ij}$.

Corollary 2 If $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is a positive semi-definite matrix, A, C are square matrices of the same order, and B is a Hermitian matrix, then $trB \leq \max\{trA, trB\}$.

Theorem3 $H = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{12}^* & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1k}^* & M_{2k}^* & \cdots & M_{kk} \end{bmatrix}$ is a positive semi-definite matrix, and $M_{11}, \ldots, M_{kk}$ are all n-order square matrices, if it satisfied M_{ij} are all Hermitian matrices, for($1 \leq i < j \leq n$),then for these $1 \leq i \leq n$, we have

$$\frac{2}{k}\lambda_i\left(\sum_{1\leq i<j\leq n} M_{ij}\right) \leq \lambda_i(H)$$

Proof Take a unitary matrix $U = \frac{1}{\sqrt{k}}\begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix}$, then U satisfied $U^*U = I_n$, noticed that M_{ij} are all Hermitian matrices, for ($1 \leq i < j \leq n$),so we have

$$U^*HU = \frac{1}{k}\sum_{i=1}^{k} M_{ii} + \frac{2}{k}\sum_{1\leq i<j\leq n} M_{ij}, \geq \frac{1}{k}\frac{2}{k-1}\sum_{1\leq i<j\leq n} M_{ij} + \frac{2}{k}\sum_{1\leq i<j\leq n} M_{ij} \quad \text{(from Theorem 2)}$$

$$= \frac{2}{k}\sum_{1\leq i<j\leq n} M_{ij}\left(\frac{1}{k-1}+1\right) \geq \frac{2}{k}\sum_{1\leq i<j\leq n} M_{ij}.$$

Then for $1 \leq i \leq n$, from lemma 1.5 we have $\frac{2}{k}\lambda_i\left(\sum_{1\leq i<j\leq n} M_{ij}\right) \leq \lambda_i(U^*HU) \leq \lambda_i(H)$, thus the conclusion $\frac{2}{k}\lambda_i\left(\sum_{1\leq i<j\leq n} M_{ij}\right) \leq \lambda_i(H)$ is proved.

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