International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 7, Issue 7, 2019, PP 12-14
ISSN No. (Print) 2347-307X & ISSN No. (Online) 2347-3142
DOI: http://dx.doi.org/10.20431/2347-3142.0707004
www.arcjournals.org

Note A Matrix Trace Inequality

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Abstract: In this note, we generalized the matrix trace inequality which is given by Yang, and the following result $|tr(A_1A_2...A_k)^m| \le [tr(A_1^{km})tr(A_2^{km})tr(A_k^{km})]^{\frac{1}{k}}$ is established, where m, k are positive integers, $A_1,\ldots, A_k$ are positive semidefinite matrix of the same order.

Keywords: *trace inequality positive semidefinite matrix*

1. Introduction

In paper [1],yang improves Bellmann's result and get the trace inequalities: For $n = 1, 2, ...$

$$0 \le tr(AB)^{2n} \le (trA)^2(trA^2)^{n-1}(trB^2)^n$$

$$0 \le tr(AB)^{2n+1} \le (trA)(trB)(trA^2)^n(trB^2)^n$$

A, B are positive semidefinite matrices of the same order.

Next, yang[4] improves this inequality: A, B are positive semidefinite matrices of the same order, then for any positive integer m, $tr(AB)^{2m} \le \left\{tr(A)^{2m}tr(B)^{2m}\right\}^{\frac{1}{2}}$.The purpose of this paper is to generalized the above matrix trace inequalites. we start with the following lemmas:

The vector of eigenvalues of A is denoted by $\lambda(A) = (\lambda_1(A), \lambda_2(A), ..., \lambda_n(A))$ If A is Hermitian, we arrange the eigenvalues of A in nonincreasing order, $\lambda_1(A) \ge \lambda_2(A) \ge ... \ge \lambda_n(A)$ .In addition $x \prec y$ means that $x = (x_1, x_2, ..., x_n)$ is majorized by $y = (y_1, y_2, ..., y_n)$ with $x_1 \ge x_2 \ge ... \ge x_n$ and $y_1 \ge y_2 \ge ... \ge y_n$, if we have $\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i$ $(k = 1, ..., n-1)$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. $x \prec_w y$ means that $x = (x_1, x_2, ..., x_n)$ is weak majorized by $y = (y_1, y_2, ..., y_n)$, if we have $\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i$ $(k = 1, ..., n)$.

Lemma1[3] Let a_{ij} $(i = 1, ..., n;\ j = 1, ..., m)$ be nonnegative real numbers, and $b_1, \ldots, b_m$ are positive integers with $\frac{1}{b_1}+\ldots+\frac{1}{b_m}=1$. Then $\sum_{i=1}^{n} a_{i1}\ldots a_{im} \le (\sum_{i=1}^{n} a_{i1}^{b_1})^{\frac{1}{b_1}} \ldots (\sum_{i=1}^{n} a_{im}^{b_m})^{\frac{1}{b_m}}$.

Lemma2[6] $A_1, \ldots, A_m$ are n -square complex matrices, then $s(\prod_{j=1}^{m} A_j) \prec_w \left\{\prod_{j=1}^{m} s_i(A_j)\right\}_{i=1}^{n}$.

Lemma3[3] Suppose $f(t)$ is a monotonically increasing function, then $x \prec_w y$ contains $(f(x_1),\ldots, f(x_n)) \prec_w (f(y_1),\ldots, f(y_n))$.

Lemma4[5] Denoted the eigenvalues of matrix A by $\lambda(A)=(\lambda_1,\ldots,\lambda_n)$, then $\left\{|\lambda_i|\right\}_{i=1}^{n} \prec_w s(A)$.

Specially , $|trA| \le \sum_{i=1}^{n} s_i(A)$.

2 Main results

Theorem5 $A_1,\ldots, A_k$ are positive semidefinite matrix of order n, then for positive integer m , k

$$|tr(A_1A_2\ldots A_k)^m| \le [tr(A_1^{km})tr(A_2^{km})tr(A_k^{km})]^{\frac{1}{k}}$$

Proof According to Lemma4, it's obvious that

$$|tr(A_1A_2\ldots A_k)^m| = |\sum_{i=1}^{n} \lambda_i(A_1A_2\ldots A_k)^m| \le \sum_{i=1}^{n} |\lambda_i(A_1A_2\ldots A_k)^m| \le \sum_{i=1}^{n} s_i(A_1A_2\ldots A_k)^m$$

Next we use Lemma1-3 can prove

$$|tr(A_1A_2\ldots A_k)^m| \le \sum_{i=1}^{n} s_i^{\,m}(A_1)s_i^{\,m}(A_2)\cdots s_i^{\,m}(A_k)$$

and

$$\sum_{i=1}^{n} s_i^{\,m}(A_1)s_i^{\,m}(A_2)\cdots s_i^{\,m}(A_k) \le \left\{\sum_{i=1}^{n} s_i^{\,km}(A_1)\sum_{i=1}^{n} s_i^{\,km}(A_2)\cdots \sum_{i=1}^{n} s_i^{\,km}(A_k)\right\}^{\frac{1}{k}}$$

On the other hand, we notice $s_i^{\,km}(A_i)=\lambda_i(A_i^{\,km})$, $(1 \le i \le n)$. Therefore

$$\left\{\sum_{i=1}^{n} s_i^{\,km}(A_1)\sum_{i=1}^{n} s_i^{\,km}(A_2)\cdots \sum_{i=1}^{n} s_i^{\,km}(A_k)\right\}^{\frac{1}{k}} = [tr(A_1)^{km}tr(A_2)^{km}tr(A_k)^{km}]^{\frac{1}{k}}.$$

So we can get

$$|tr(A_1A_2\ldots A_k)^m| \le [tr(A_1^{km})tr(A_2^{km})tr(A_k^{km})]^{\frac{1}{k}}$$

Corollary6 [4] A, B are positive semidefinite matrices of the same order, then for positive integer m ,

$$tr(AB)^m \le [tr(A)^{2m}tr(B)^{2m}]^{\frac{1}{2}}$$

Corollary7[1] A, B are positive semidefinite matrices of the same order, then for $n=1,2,\ldots$

$$0 \le tr(AB)^{2n} \le (trA)^2(trA^2)^{n-1}(trB^2)^n$$

$$0 \le tr(AB)^{2n+1} \le (trA)(trB)(trA^2)^n (trB^2)^n$$

Corollary8 $A_1, \ldots A_k$ are positive semidefinite matrix of order n, then for positive integer m , k

$$|tr(A_1 A_2 \ldots A_k)^m| \le tr(A_1{}^m) tr(A_2{}^m) tr(A_k{}^m)$$

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Citation: *Feng Zhang, et.al., (2019). Note A Matrix Trace Inequality. International Journal of Scientific and Innovative Mathematical Research (IJSIMR), 7(7), pp. 12-14. http://dx.doi.org/ 10.20431/2347 -3142.0707004*