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An Inequalities for the Trace of the Block Hadamard Product

Hui Quan*

Department of mathematics, Xiangtan University, Xiangtan 411105, China

Corresponding Author:** ***Hui Quan, *Department of mathematics, Xiangtan University, Xiangtan 411105, China*

Abstract: *Let* $\lambda_1(A) \geq ... \geq \lambda_n(A)$ *denote the eigenvalues of a Hermitian* n *by* n *matrix* A, *our main results are* $\log \lambda(C^*C) \prec_w \log \lambda(A) \circ \lambda(B)$ *and* $\lambda(C^*C) \prec_w \lambda(A) \circ \lambda(B)$. *Here* A *is a positive matrix,* B *and* $M = \begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ *are positive semi-definite Hermitian matrices.*

Keywords: *Eigenvalue, Positive Semi-definite Matrix, Majorization*

1. Introduction

Let A and B be positive semi-definite. The following results due to Boying Wang and Fuzhen Zhang [1] says that $\prod_{j=1}^{k} \lambda_t(AB) \geq \prod_{j=1}^{k} \lambda_{i_t}(A)\lambda_{n-i_t+1}(B)$ with $1 \leq i_1 < ... < i_k \leq n$. Gunther and Klotz presented a survey focusing on the study of a block Hadamard and block Kronecker products of positive semidefinite matrices in 2012[3]. Saliha Pehlivan [4] provided some trace inequalities for the trace of the block Hadamard product. In this paper, we shall be mainly interested in the inequalities for the trace of the block Hadamard product.

2. Preliminaries

Let $C^{n\times n}$ denote the vector space of all n by n complex matrices. Denote by $\lambda_1(A) \geq ... \geq \lambda_n(A)$ the eigenvalues of a Hermitian matrix $A \in C^{n\times n}$.

Let $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$ with $x_1 \geq x_2 \geq ... \geq x_n$, $y_1 \geq y_2 \geq ... \geq y_n$ .If $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, $k = 1, 2, ..., n$, we say that x is weakly majorized by y ,and write $x \prec_w y$.

If $\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i$, $k = 1, 2, ..., n$, we write $x \prec_{w\log} y$,or $\log x \prec_w \log y$.

3. Main Results

We start with the following known results.

Lemma1.1 If $A, B \in C^{n\times n}$ be *Hermitian*, then for any $1 \leq j \leq n$,

we have $\max_{r+s=j+n} \{\lambda_r(A) + \lambda_s(B)\} \leq \lambda_j(A+B) \leq \min_{r+s=j+1} \{\lambda_r(A) + \lambda_s(B)\}$.

Lemma1.2 If A, B are the same order matrix, the AB and BA have the same non-zero eigenvalues. Similarly, the same order matrix A, B, C, ABC and CAB also have the same non-zero eigenvalues.

Lemma1.3 If $A, B \in C^{n\times n}$ are positive semi-definite Hermitian and $1 \leq i_1 < ... < i_k \leq n$,then

$\prod_{j=1}^{k}\lambda_{t}(AB)\ge\prod_{j=1}^{k}\lambda_{i_t}(A)\lambda_{n-i_t+1}(B)$ with equality for $k=n$.

Lemma1.4 For $x,y\in R_{+}^{n}$, $\prod_{j=1}^{k}x_j^{\downarrow}\le\prod_{j=1}^{k}y_j^{\downarrow},1\le k\le n\Rightarrow\sum_{j=1}^{k}x_j^{\downarrow}\le\sum_{j=1}^{k}y_j^{\downarrow},1\le k\le n$. Where $x_j^{\downarrow},y_j^{\downarrow}$ means reorder $x_1,x_2,\cdots,x_k$ and $y_1,y_2,\cdots,y_k$ from smallest to largest.

Theorem 1 A,B,C are square matrices of order n, with A being positive definite matrix, B and $M=\begin{bmatrix}A & C^{*}\\ C & B\end{bmatrix}$ are positive semi-definite matrices,

then $\log\lambda(C^{*}C)\prec_{w}\log\lambda(A)\circ\lambda(B)$, $\lambda(C^{*}C)\prec_{w}\lambda(A)\circ\lambda(B)$.

Proof Considered contract changes

$$\begin{bmatrix}I & -A^{-1}C^{*}\\ 0 & I\end{bmatrix}^{*}\begin{bmatrix}A & C^{*}\\ C & B\end{bmatrix}\begin{bmatrix}I & -A^{-1}C^{*}\\ 0 & I\end{bmatrix}=\begin{bmatrix}A & 0\\ 0 & B-CA^{-1}C^{*}\end{bmatrix}$$

Due to $M=\begin{bmatrix}A & C^{*}\\ C & B\end{bmatrix}$ is a positive definite matrix, from the properties of positive definite matrix we can know that $B-CA^{-1}C^{*}\ge 0$, in other words $B\ge CA^{-1}C^{*}$, so, from lemma 1.1 we have $\lambda_j(B)\ge\lambda_j(CA^{-1}C^{*})$, from lemma 1.2 we have

$\lambda_j(CA^{-1}C^{*})=\lambda_j(C^{*}CA^{-1})$, thus $\lambda_j(B)\ge\lambda_j(C^{*}CA^{-1})$. from lemma 1.3 we know that

$$\prod_{j=1}^{k}\lambda_j(B)\ge\prod_{j=1}^{k}\lambda_j(C^{*}CA^{-1})\ge\prod_{j=1}^{k}\lambda_j(C^{*}C)\lambda_{n-j+1}(A^{-1}),1\le k\le n.$$

Due to $\lambda_{n-j+1}(A^{-1})=\dfrac{1}{\lambda_j(A)}$, we can know $\prod_{j=1}^{k}\lambda_j(B)\lambda_j(A)\ge\prod_{j=1}^{k}\lambda_j(C^{*}C),1\le k\le n$. So we can prove that $\lambda(C^{*}C)\prec_{w\log}\lambda(A)\circ\lambda(B)$. Obviously $\prod_{j=1}^{k}\lambda_j(B)\lambda_j(A)=\prod_{j=1}^{k}\lambda^{\downarrow}{}_j(B)\lambda^{\downarrow}{}_j(A)$

$\prod_{j=1}^{k}\lambda_j(C^{*}C)=\prod_{j=1}^{k}\lambda^{\downarrow}{}_j(C^{*}C)$, then we have $\prod_{j=1}^{k}\lambda^{\downarrow}{}_j(B)\lambda^{\downarrow}{}_j(A)\ge\prod_{j=1}^{k}\lambda^{\downarrow}{}_j(C^{*}C),1\le k\le n$. from lemma 1.4 we know that $\sum_{j=1}^{k}\lambda^{\downarrow}{}_j(A)\lambda^{\downarrow}{}_j(B)\ge\sum_{j=1}^{k}\lambda^{\downarrow}{}_j(C^{*}C),1\le k\le n$. So we can get that $\lambda(C^{*}C)\prec_{w}\lambda(A)\circ\lambda(B)$.

From $\lambda(C^{*}C)\prec_{w\log}\lambda(A)\circ\lambda(B)$ we can get that $\lambda(C^{*}C)^{p}\prec_{w\log}\lambda(A^{p})\circ\lambda(B^{p})$, where $0<p<1$.

In particular, when $p=\dfrac{1}{2}$, we can also get $\delta(C)\prec_{w\log}\lambda(A^{\frac{1}{2}})\circ\lambda(B^{\frac{1}{2}})$, and then we can get

$\delta(C)\prec_{w}\lambda(A^{\frac{1}{2}})\circ\lambda(B^{\frac{1}{2}})$. When $k=n$, $\prod_{j=1}^{n}\lambda_j(B)\lambda_j(A)\ge\prod_{j=1}^{n}\lambda_j(C^{*}C)$, we can know that $\det A\det B\ge\det(C^{*}C)$.

Note

we know that when $A,B\ge 0$, $\lambda(A\circ B)\prec_{w}\lambda(A)\circ\lambda(B)$ is true, but the inequality in the above theorem cannt be strengthened to $\lambda(C^{*}C)\prec_{w}\lambda(A\circ B)$.

Here is a example：

Take $A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}, B=\begin{bmatrix}3 & 2\\ 2 & 1.6\end{bmatrix}, C=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$, it can be calculated that

$tr(C^*C) \ge tr(A\text{o}B)$.

Corollary 1 A, B, C, M conditions are the same as above, the $trAtrB \ge tr(C^*C)$.

Corollary 2 Let A, B are n-order Hermitematrix, and the block matrix $\begin{bmatrix}A & C^*\\ C & B\end{bmatrix} \ge 0$, $B \ge 0$, then $\delta(C)\ \text{p}_{w\log}\ u$, where $u=(u_1,u_2,...,u_n)$, $u_i = \max\{\lambda_i(A), \lambda_i(B)\}$.

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